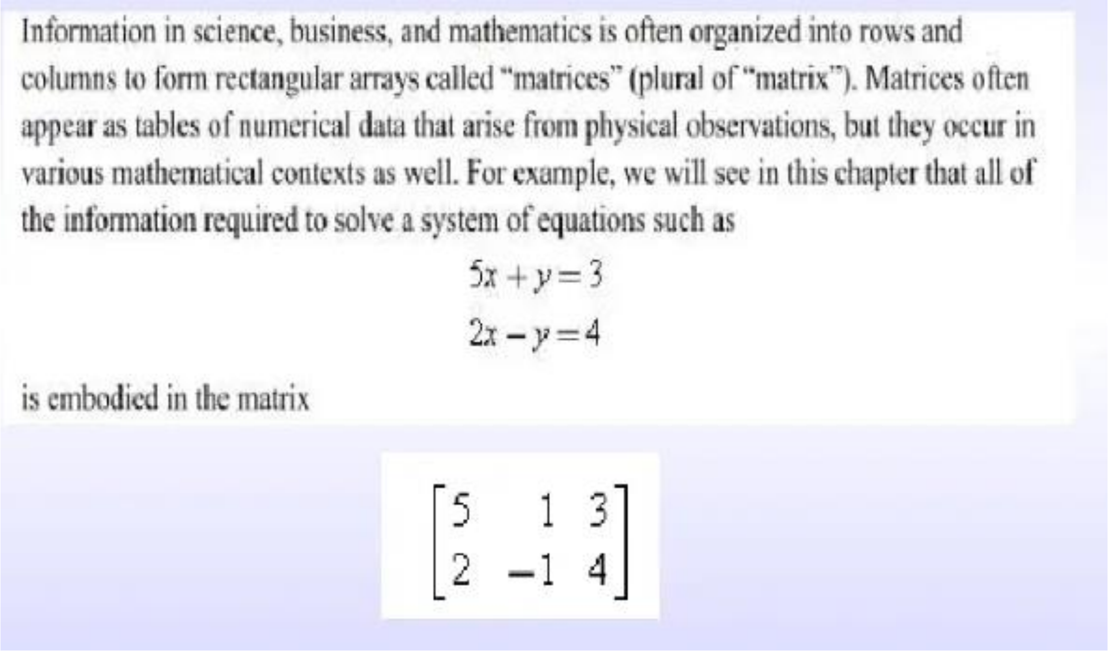
CHAPTER 1: Systems of Linear Equations and Matrices

1.1 Introduction to Systems of Linear Equations



Information in science, business, and mathematics is often organized into rows and columns to form rectangular arrays called "matrices" (plural of "matrix"). Matrices often appear as tables of numerical data that arise from physical observations, but they occur in various mathematical contexts as well. For example, we will see in this chapter that all of the information required to solve a system of equations such as

$$5x + y = 3$$
$$2x - y = 4$$

is embodied in the matrix

$$\begin{bmatrix} 5 & 1 & 3 \\ 2 & -1 & 4 \end{bmatrix}$$

Linear Equations Recall that in two dimensions a line in a rectangular *xy*-coordinate system can be represented by an equation of the form

$ax + by =c$ (a,b not both 0)

and in three dimensions a plane in a rectangular xyz-coordinate system can be represented by an equation of the form

$ax + by + cz =d$ (a,b,c not all 0)

These are examples of "linear equations," the first being a linear equation in the variables x and y and the second a linear equation in the variables x, y, and z. More generally, we define a ***linear equation*** in the n variables $x_1, x_2, \ldots, x_n$ to be one that can be expressed in the form

$a_1x_1 + a_2x_2 +\cdot\cdot\cdot+a_nx_n = b$ (1)

where $a_1, a_2, \ldots, a_n$ and b are constants, and the a's are not all zero. In the special cases where $n = 2$ or $n = 3$, we will often use variables without subscripts and write linear equations as

$a_1x + a_2y = b \qquad (a_1, a_2 \text{ not both } 0) \qquad (2)$

$a_1x + a_2y + a_3z = b \; (a_1, a_2, a_3 \text{ not all } 0) \qquad (3)$

In the special case where $b = 0$, Equation (1) has the form

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \qquad (4)$

which is called a ***homogeneous linear equation*** in the variables $x_1, x_2, \ldots, x_n$.

EXAMPLE 1

Observe that a linear equation does not involve any products or roots of variables. All variables occur only to the first power and do not appear, for example, as arguments of trigonometric, logarithmic, or exponential functions. The following are linear equations:

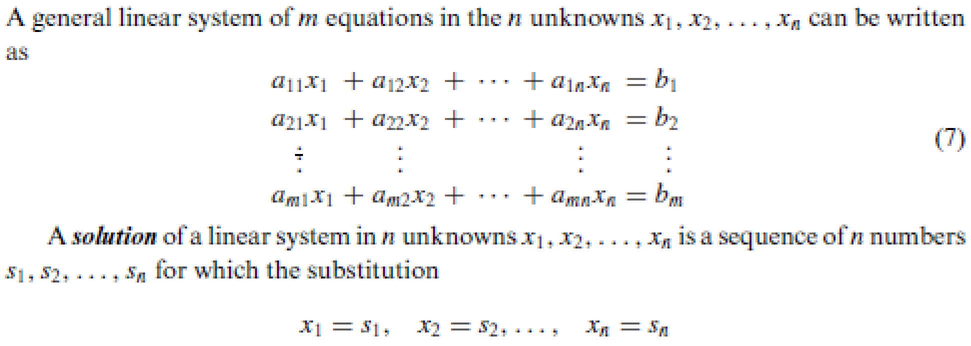
$$\begin{array}{ll} x + 3y = 7 & x_1 - 2x_2 - 3x_3 + x_4 = 0 \\ \frac{1}{2}x - y + 3z = -1 & x_1 + x_2 + \cdots + x_n = 1 \end{array}$$

The following are not linear equations:

$$\begin{array}{ll} x + 3y^2 = 4 & 3x + 2y - xy = 5 \\ \sin x + y = 0 & \sqrt{x_1} + 2x_2 + x_3 = 1 \end{array}$$

A finite set of linear equations is called a ***system of linear equations*** or, more briefly, a ***linear system***. The variables are called ***unknowns***. For example, system (5) that follows has unknowns x and y, and system (6) has unknowns x_1, x_2, and x_3.

$$\begin{array}{ll} 5x + y = 3 & 4x_1 - x_2 + 3x_3 = -1 \\ 2x - y = 4 & 3x_1 + x_2 + 9x_3 = -4 \end{array} \qquad (5\text{–}6)$$



A general linear system of m equations in the n unknowns $x_1, x_2, \ldots, x_n$ can be written as

$$\begin{array}{ccccccc} a_{11}x_1 & + a_{12}x_2 & + & \cdots & + a_{1n}x_n & = b_1 \\ a_{21}x_1 & + a_{22}x_2 & + & \cdots & + a_{2n}x_n & = b_2 \\ \vdots & \vdots & & & \vdots & \vdots \\ a_{m1}x_1 & + a_{m2}x_2 & + & \cdots & + a_{mn}x_n & = b_m \end{array} \qquad (7)$$

A ***solution*** of a linear system in n unknowns $x_1, x_2, \ldots, x_n$ is a sequence of n numbers $s_1, s_2, \ldots, s_n$ for which the substitution

$$x_1 = s_1, \quad x_2 = s_2, \ldots, \quad x_n = s_n$$

Linear Systems in Two and Three Unknowns

Linear systems in two unknowns arise in connection with intersections of lines. For example, consider the linear system

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

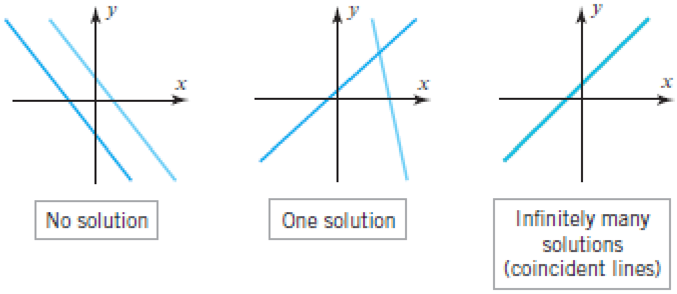
in which the graphs of the equations are lines in the xy-plane. Each solution (x, y) of this system corresponds to a point of intersection of the lines, so there are three possibilities:

1. The lines may be parallel and distinct, in which case there is no intersection and consequently no solution.

2. The lines may intersect at only one point, in which case the system has exactly one solution.

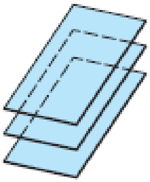
3. The lines may coincide, in which case there are infinitely many points of intersection (the points on the common line) and consequently infinitely many solutions.

In general, we say that a linear system is **consistent** if it has at least one solution and **inconsistent** if it has no solutions. Thus, a *consistent* linear system of two equations in two unknowns has either one solution or infinitely many solutions—there are no other possibilities. The same is true for a linear system of three equations in three unknowns

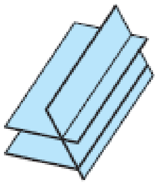


$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

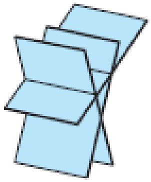
in which the graphs of the equations are planes. The solutions of the system, if any, correspond to points where all three planes intersect, so again we see that there are only three possibilities—no solutions, one solution, or infinitely many solutions



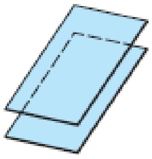
No solutions
(three parallel planes;
no common intersection)



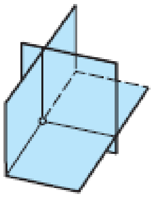
No solutions
(two parallel planes;
no common intersection)



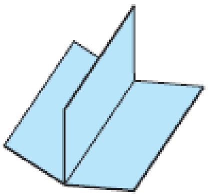
No solutions
(no common intersection)



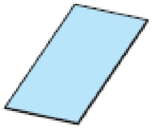
No solutions
(two coincident planes
parallel to the third;
no common intersection)



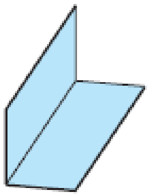
One solution
(intersection is a point)



Infinitely many solutions
(intersection is a line)



Infinitely many solutions
(planes are all coincident;
intersection is a plane)



Infinitely many solutions
(two coincident planes;
intersection is a line)

EXAMPLE 2 A Linear System with One Solution

Solve the linear system

$$x - y = 1$$
$$2x + y = 6$$

Solution We can eliminate x from the second equation by adding −2 times the first equation to the second. This yields the simplified system

$x - y = 1$

$3y = 4$

From the second equation we obtain $y = 4/3$, and on substituting this value in the first equation we obtain $x = 1 + y = 7/3$. Thus, the system has the unique solution

$x = 7/3, y = 4/3$

Geometrically, this means that the lines represented by the equations in the system intersect at the single point (7/3, 4/3).

EXAMPLE 3 A Linear System with No Solutions

Solve the linear system

$$x + y = 4$$
$$3x + 3y = 6$$

Solution We can eliminate x from the second equation by adding −3 times the first equation to the second equation. This yields the simplified system

$$x + y = 4$$
$$0 = -6$$

The second equation is contradictory, so the given system has no solution. Geometrically, this means that the lines corresponding to the equations in the original system are parallel and distinct. they have the same slope but different y-intercepts.

EXAMPLE 4 A Linear System with Infinitely Many Solutions

Solve the linear system

$$4x - 2y = 1$$
$$16x - 8y = 4$$

Solution We can eliminate x from the second equation by adding −4 times the first equation to the second. This yields the simplified system

$$4x - 2y = 1$$
$$0 = 0$$

The second equation does not impose any restrictions on x and y and hence can be omitted. Thus, the solutions of the system are those values of x and y that satisfy the single equation

$$4x - 2y = 1 \qquad (8)$$

Geometrically, this means the lines corresponding to the two equations in the original system coincide.

One way to describe the solution set is to solve this equation for x in terms of y to obtain $x = \frac{1}{4} + \frac{1}{2}y$ and then assign an arbitrary value t (called a ***parameter***)

EXAMPLE 5 A Linear System with Infinitely Many Solutions

Solve the linear system

$$x - y + 2z = 5$$
$$2x - 2y + 4z = 10$$
$$3x - 3y + 6z = 15$$

Solution This system can be solved by inspection, since the second and third equations are multiples of the first. Geometrically, this means that the three planes coincide and that those values of x, y, and z that satisfy the equation

$$x - y + 2z = 5 \qquad (9)$$

automatically satisfy all three equations. Thus, it suffices to find the solutions of (9).

We can do this by first solving this equation for x in terms of y and z, then assigning arbitrary values r and s (parameters) to these two variables, and then expressing the solution by the three parametric equations

$x = 5 + r - 2s,\ y = r,\ z = s$

Specific solutions can be obtained by choosing numerical values for the parameters r and s. For example, taking r = 1 and s = 0 yields the solution (6, 1, 0).

The basic method for solving a linear system is to perform algebraic operations on the system that do not alter the solution set and that produce a succession of increasingly simpler systems, until a point is reached where it can be ascertained whether the system is consistent, and if so, what its solutions are.

Typically, the algebraic operations are:

1. Multiply an equation through by a nonzero constant.

2. Interchange two equations.

3. Add a constant times one equation to another.

Since the rows (horizontal lines) of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operations on the rows of the augmented matrix:

1. Multiply a row through by a nonzero constant.

2. Interchange two rows.

3. Add a constant times one row to another.

These are called ***elementary row operations*** on a matrix.

In the following example we will illustrate how to use elementary row operations and an augmented matrix to solve a linear system in three unknowns. Since a systematic procedure for solving linear systems will be developed in the next section, do not worry about how the steps in the example were chosen. Your objective here should be simply to understand the computations.

EXAMPLE 6 Using Elementary Row Operations

In the left column we solve a system of linear equations by operating on the equations in the system, and in the right column we solve the same system by operating on the rows of the augmented matrix.

$$\begin{aligned} x + y + 2z &= 9 \\ 2x + 4y - 3z &= 1 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add − 2 times the first equation to the second to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3x + 6y - 5z &= 0 \end{aligned}$$

Add − 2 times the first row to the second to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{bmatrix}$$

Add -3 times the first equation to the third to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ 3y - 11z &= -27 \end{aligned}$$

Add -3 times the first row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Multiply the second equation by $\frac{1}{2}$ to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ 3y - 11z &= -27 \end{aligned}$$

Multiply the second row by $\frac{1}{2}$ to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

Add -3 times the second equation to the third to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ -\tfrac{1}{2}z &= -\tfrac{3}{2} \end{aligned}$$

Add -3 times the second row to the third to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

Multiply the third equation by -2 to obtain

$$\begin{aligned} x + y + 2z &= 9 \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Multiply the third row by -2 to obtain

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add -1 times the second equation to the first to obtain

$$\begin{aligned} x \quad + \tfrac{11}{2}z &= \tfrac{35}{2} \\ y - \tfrac{7}{2}z &= -\tfrac{17}{2} \\ z &= 3 \end{aligned}$$

Add -1 times the second row to the first to obtain

$$\begin{bmatrix} 1 & 0 & \frac{11}{2} & \frac{35}{2} \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Add $-\frac{11}{2}$ times the third equation to the first and $\frac{7}{2}$ times the third equation to the second to obtain

$$\begin{aligned} x \qquad &= 1 \\ y \quad &= 2 \\ z &= 3 \end{aligned}$$

Add $-\frac{11}{2}$ times the third row to the first and $\frac{7}{2}$ times the third row to the second to obtain

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The solution $x = 1$, $y = 2$, $z = 3$ is now evident. ◀

1.2 Matrices and Matrix Operations

DEFINITION 1 A ***matrix*** is a rectangular array of numbers. The numbers in the array are called the ***entries*** in the matrix.

EXAMPLE 1 Examples of Matrices

Some examples of matrices are

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad [2 \quad 1 \quad 0 \quad -3], \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad [4]$$

A matrix with only one row, such as the second in Example 1, is called a ***row vector***

(or a ***row matrix***), and a matrix with only one column, such as the fourth in that example ,is called a ***column vector*** (or a ***column matrix***). The fifth matrix in that example is both a row vector and a column vector.

We will use capital letters to denote matrices and lowercase letters to denote numerical quantities.

The ***size*** of a matrix is described in terms of the number of rows (horizontal lines) and columns (vertical lines) it contains.

Thus a general 3 × 4 matrix might be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

the first notation being used when it is important in the discussion to know the size, and the second when the size need not be emphasized. Usually, we will match the letter denoting a matrix with the letter denoting its entries; thus, for a matrix *B* we would generally use b_{ij} for the entry in row *i* and column *j* , and for a matrix *C* we would use the notation c_{ij} .

The entry in row *i* and column *j* of a matrix *A* is also commonly denoted by the symbol $(A)_{ij}$. Thus, for matrix (1) above, we have $(A)_{ij} = a_{ij}$ and for the matrix

$$A = \begin{bmatrix} 2 & -3 \\ 7 & 0 \end{bmatrix}$$

we have $(A)_{11} = 2$, $(A)_{12} = -3$, $(A)_{21} = 7$, and $(A)_{22} = 0$.

A matrix *A* with *n* rows and *n* columns is called a ***square matrix of order n***, and the shaded entries $a_{11}, a_{22}, \ldots, a_{nn}$ in (2) are said to be on the ***main diagonal*** of *A*.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad (2)$$

DEFINITION 2 Two matrices are defined to be ***equal*** if they have the same size and their corresponding entries are equal.

EXAMPLE 2 Equality of Matrices

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

If $x = 5$, then $A = B$, but for all other values of *x* the matrices *A* and *B* are not equal, since not all of their corresponding entries are equal. There is no value of *x* for which $A = C$ since *A* and *C* have different sizes.

DEFINITION 3 If *A* and *B* are matrices of the same size, then the ***sum*** $A + B$ is the matrix obtained by adding the entries of *B* to the corresponding entries of *A*, and the ***difference*** $A - B$ is the matrix obtained by subtracting the entries of *B* from the corresponding entries of *A*. Matrices of different sizes cannot be added or subtracted.

EXAMPLE 3 **Addition and Subtraction**

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \quad \text{and} \quad A - B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

The expressions $A + C$, $B + C$, $A - C$, and $B - C$ are undefined.

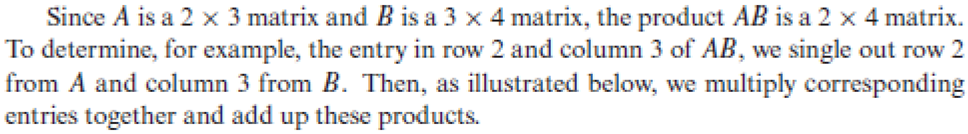
DEFINITION 4 If A is any matrix and c is any scalar, then the ***product*** cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is said to be a ***scalar multiple*** of A.

DEFINITION 5 If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the ***product*** AB is the $m \times n$ matrix whose entries are determined as follows: To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together, and then add up the resulting products.

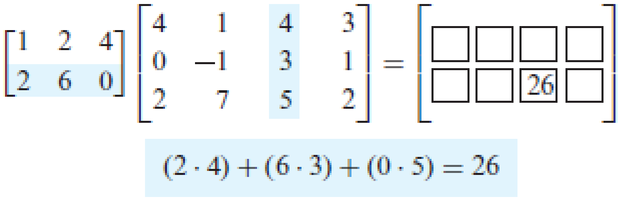
EXAMPLE 5 Multiplying Matrices

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$



Since A is a 2×3 matrix and B is a 3×4 matrix, the product AB is a 2×4 matrix. To determine, for example, the entry in row 2 and column 3 of AB, we single out row 2 from A and column 3 from B. Then, as illustrated below, we multiply corresponding entries together and add up these products.



$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{26} & \boxed{} \end{bmatrix}$$

$$(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26$$

The entry in row 1 and column 4 of AB is computed as follows:

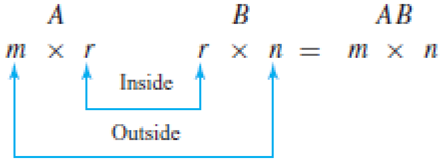
$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}\begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \boxed{13} \\ \square & \square & \square & \square \end{bmatrix}$$

$$(1 \cdot 3) + (2 \cdot 1) + (4 \cdot 2) = 13$$

The computations for the remaining entries are

$$\begin{aligned} (1 \cdot 4) + (2 \cdot 0) + (4 \cdot 2) &= 12 \\ (1 \cdot 1) - (2 \cdot 1) + (4 \cdot 7) &= 27 \\ (1 \cdot 4) + (2 \cdot 3) + (4 \cdot 5) &= 30 \\ (2 \cdot 4) + (6 \cdot 0) + (0 \cdot 2) &= 8 \\ (2 \cdot 1) - (6 \cdot 1) + (0 \cdot 7) &= -4 \\ (2 \cdot 3) + (6 \cdot 1) + (0 \cdot 2) &= 12 \end{aligned} \qquad AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

The outside numbers then give the size of the product.



EXAMPLE 6 Determining Whether a Product Is Defined

Suppose that *A*, *B*, and *C* are matrices with the following sizes:

A	*B*	*C*
3 × 4	4 × 7	7 × 3

AB is defined and is a 3 × 7 matrix; *BC* is defined and is a 4 × 3 matrix; and *CA* is defined and is a 7 × 4 matrix. The products *AC*,*CB*, and *BA* are all undefined.

Partitioned Matrices A matrix can be subdivided or ***partitioned*** into smaller matrices by inserting horizontal and vertical rules between selected rows and columns. For example, the following are three possible partitions of a general 3 × 4 matrix *A*—the first is a partition of *A* into four ***sub matrices*** A_{11},A_{12},A_{21}, andA_{22}; the second is a partition of *A* into its row vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$; and the third is a partition of *A* into its column vectors $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$, and $\mathbf{c}_4$:

$$A = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}$$

$$A = \left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right] = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3 \quad \mathbf{c}_4]$$

Matrix Multiplication by Columns and by Rows

$$AB = A[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n] = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_n]$$

(*AB* computed column by column)

$$AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B \end{bmatrix}$$

(*AB* computed row by row)

In words, these formulas state that

$$j\text{th column vector of } AB = A[\, j\text{th column vector of } B]$$

$$i\text{th row vector of } AB = [i\text{th row vector of } A]B$$

EXAMPLE 7 Example 5 Revisited

If *A* and *B* are the matrices in Example 5, the second column vector of *AB* can be obtained by the computation

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \underset{\substack{\uparrow \\ \text{Second column} \\ \text{of } B}}{\begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}} = \underset{\substack{\uparrow \\ \text{Second column} \\ \text{of } AB}}{\begin{bmatrix} 27 \\ -4 \end{bmatrix}}$$

the first row vector of *AB* can be obtained by the computation

$$\underset{\text{First row of } A}{\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \underset{\text{First row of } AB}{\begin{bmatrix} 12 & 27 & 30 & 13 \end{bmatrix}}$$

To see how matrix products can be viewed as linear combinations, let *A* be an *m* × *n* matrix and **x** an *n* × 1 column vector, say

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

EXAMPLE 8 Matrix Products as Linear Combinations

The matrix product

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

can be written as the following linear combination of column vectors:

$$2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

EXAMPLE 9 Columns of a Product *AB* as Linear Combinations

We showed in Example 5 that

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 27 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ 26 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 6 \end{bmatrix} + 5\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 13 \\ 12 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Matrix Form of a Linear System

Matrix multiplication has an important application to systems of linear equations. Consider a system of m linear equations in n unknowns:

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m \end{array}$$

Since two matrices are equal if and only if their corresponding entries are equal, we can replace the m equations in this system by the single matrix equation

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The $m \times 1$ matrix on the left side of this equation can be written as a product to give

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If we designate these matrices by A, $\mathbf{x}$, and $\mathbf{b}$, respectively, then we can replace the original system of m equations in n unknowns by the single matrix equation

$$A\mathbf{x} = \mathbf{b}$$

The matrix *A* in this equation is called the ***coefficient matrix*** of the system. The augmented matrix for the system is obtained by adjoining **b** to *A* as the last column; thus the augmented matrix is

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

DEFINITION 7 If A is any $m \times n$ matrix, then the ***transpose of A***, denoted by A^T, is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A; that is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth.

EXAMPLE 11 Some Transposes

The following are some examples of matrices and their transposes.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C = [1 \quad 3 \quad 5], \quad D = [4]$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad D^T = [4] \blacktriangleleft$$

DEFINITION 8 If A is a square matrix, then the ***trace of A***, denoted by $\operatorname{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

EXAMPLE 12 Trace

The following are examples of matrices and their traces.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\operatorname{tr}(A) = a_{11} + a_{22} + a_{33} \qquad \operatorname{tr}(B) = -1 + 5 + 7 + 0 = 11$$

Exercises Use the following matrices to compute the indicated expression if it is defined.

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

(a) $D + E$
(b) $D - E$
(c) $5A$
(d) $-7C$
(e) $2B - C$
(f) $4E - 2D$
(g) $-3(D + 2E)$
(h) $A - A$
(i) $\mathrm{tr}(D)$
(j) $\mathrm{tr}(D - 3E)$
(k) $4\,\mathrm{tr}(7B)$
(l) $\mathrm{tr}(A)$

Exercises Find matrices *A*, **x**, and **b** that express the given linear system as a single matrix equation *A***x** = **b**, and write out this matrix equation.

(a)
$$\begin{aligned} 2x_1 - 3x_2 + 5x_3 &= 7 \\ 9x_1 - x_2 + x_3 &= -1 \\ x_1 + 5x_2 + 4x_3 &= 0 \end{aligned}$$

(b)
$$\begin{aligned} 4x_1 \qquad - 3x_3 + x_4 &= 1 \\ 5x_1 + x_2 \qquad - 8x_4 &= 3 \\ 2x_1 - 5x_2 + 9x_3 - x_4 &= 0 \\ 3x_2 - x_3 + 7x_4 &= 2 \end{aligned}$$

1.3 Inverses; Algebraic Properties of Matrices

Properties of Matrix Addition and Scalar Multiplication

THEOREM 1.3.1 Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

(a) A + B = B + A **[Commutative law for matrix addition]**

(b) A + (B + C) = (A + B) + C **[Associative law for matrix addition]**

(c) A(BC) = (AB)C **[Associative law for matrix multiplication]**

(d) A(B + C) = AB + AC **[Left distributive law]**

(e) (B + C)A = BA + CA **[Right distributive law]**

(f) A(B − C) = AB − AC

(g) (B − C)A = BA − CA

(h) a(B + C) = aB + aC

(i) a(B − C) = aB − aC

(j) (a + b)C = aC + bC

(k) (a − b)C = aC − bC

(l) a(bC) = (ab)C

(m) a(BC) = (aB)C = B(aC)

EXAMPLE 1 Associatively of Matrix Multiplication

As an illustration of the associative law for matrix multiplication, consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad BC = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix}$$

Thus

$$(AB)C = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

and

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}$$

so *(AB)C* = *A(BC)*.

In matrix arithmetic, however, the equality of *AB* and *BA* can fail for three possible reasons:

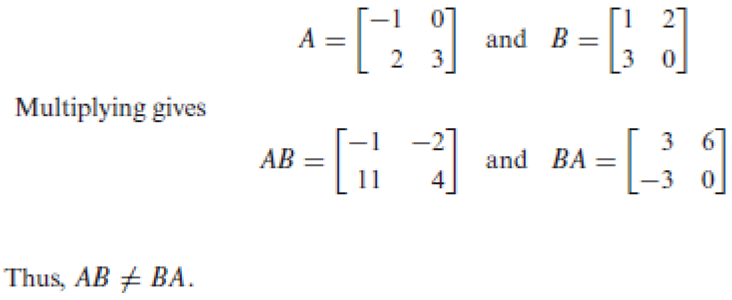
1. *AB* may be defined and *BA* may not (for example, if *A* is 2 × 3 and *B* is 3 × 4).

2. *AB* and *BA* may both be defined, but they may have different sizes (for example, if *A* is 2 × 3 and *B* is 3 × 2).

3. *AB* and *BA* may both be defined and have the same size, but the two products may be different (as illustrated in the next example).

EXAMPLE 2 Order Matters in Matrix Multiplication

Consider the matrices



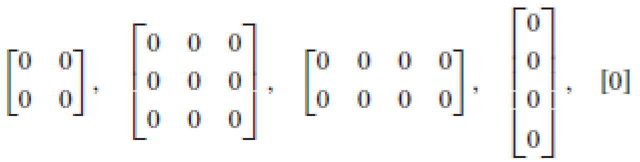
$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

Multiplying gives

$$AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

Thus, $AB \neq BA$.

Zero Matrices A matrix whose entries are all zero is called a **zero matrix**. Some examples are



$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [0]$$

THEOREM 1.3.2 Properties of Zero Matrices

If c is a scalar, and if the sizes of the matrices are such that the operations can be perfomed, then:

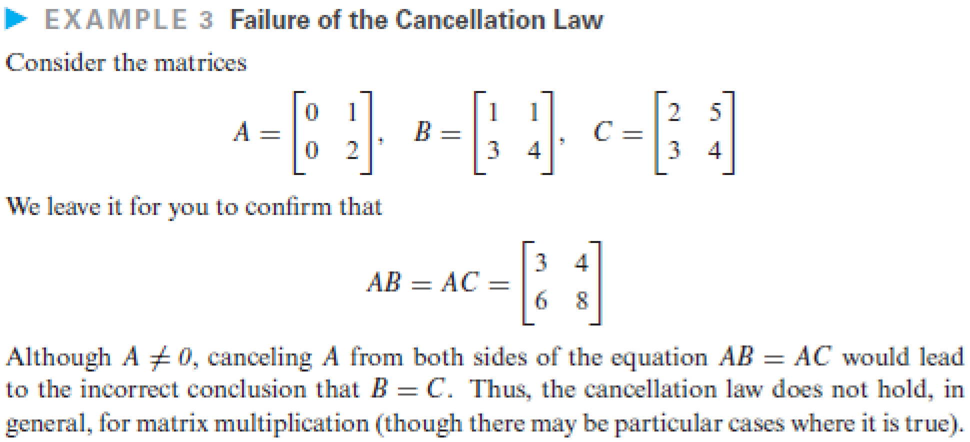
(a) A + 0 = 0 + A = A

(b) A − 0 = A

(c) A − A = A + (−A) = 0

(d) 0A = 0

(e) If cA = 0, then c = 0 or A = 0.



▶ **EXAMPLE 3 Failure of the Cancellation Law**

Consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$

We leave it for you to confirm that

$$AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

Although $A \neq 0$, canceling A from both sides of the equation $AB = AC$ would lead to the incorrect conclusion that $B = C$. Thus, the cancellation law does not hold, in general, for matrix multiplication (though there may be particular cases where it is true).

▶ EXAMPLE 4 A Zero Product with Nonzero Factors

Here are two matrices for which $AB = 0$, but $A \neq 0$ and $B \neq 0$:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$$ ◀

Identity Matrices A square matrix with 1's on the main diagonal and zeros elsewhere is called an **identity matrix**. Some examples are

$$[1], \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$AI_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

The same result holds in general; that is, if A is any m × n matrix, then

$A I_n = A$ and $I_m A = A$

Inverse of a Matrix In real arithmetic every nonzero number a has a reciprocal

$a^{-1}(= 1/a)$ with the property $a \cdot a^{-1} = a^{-1} \cdot a = 1$

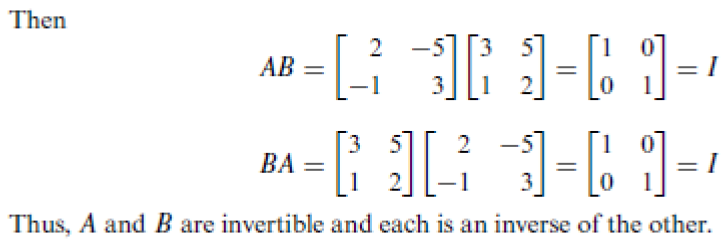
The number a^{-1} is sometimes called the *multiplicative inverse* of a. Our next objective is to develop an analog of this result for matrix arithmetic. For this purpose we make the following definition.

DEFINITION 1 If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be ***invertible*** (or ***nonsingular***) and B is called an ***inverse*** of A. If no such matrix B can be found, then A is said to be ***singular***.

EXAMPLE 5 An Invertible Matrix

Let

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

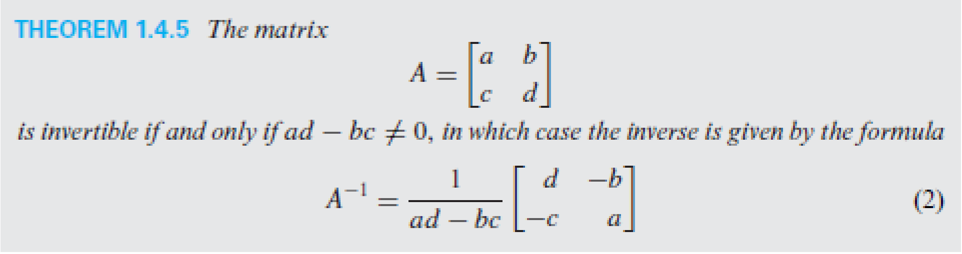


Then

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, A and B are invertible and each is an inverse of the other.

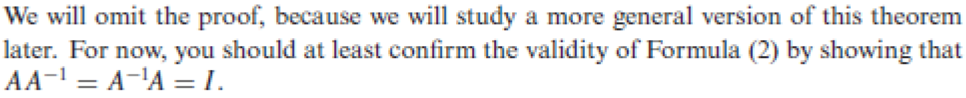


THEOREM 1.4.5 *The matrix*

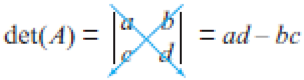
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, *in which case the inverse is given by the formula*

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{2}$$



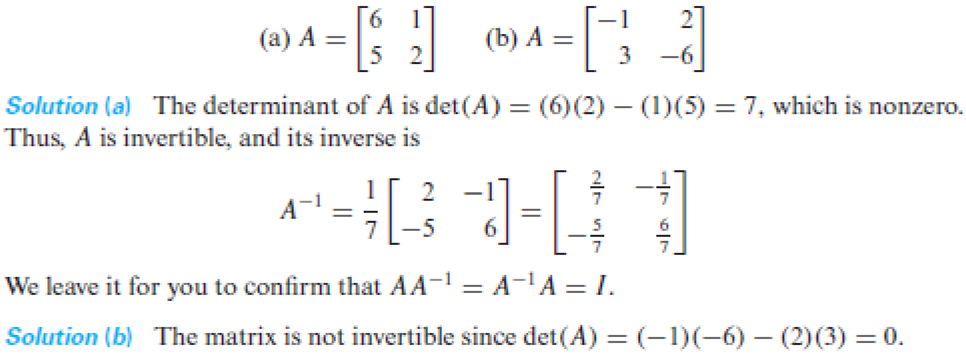
We will omit the proof, because we will study a more general version of this theorem later. For now, you should at least confirm the validity of Formula (2) by showing that $AA^{-1} = A^{-1}A = I$.



$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

EXAMPLE 7 Calculating the Inverse of a 2 × 2 Matrix

In each part, determine whether the matrix is invertible. If so, find its inverse.



$$\text{(a) } A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \qquad \text{(b) } A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$$

Solution (a) The determinant of A is $\det(A) = (6)(2) - (1)(5) = 7$, which is nonzero. Thus, A is invertible, and its inverse is

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

We leave it for you to confirm that $AA^{-1} = A^{-1}A = I$.

Solution (b) The matrix is not invertible since $\det(A) = (-1)(-6) - (2)(3) = 0$.

THEOREM 1.3.6 *If A and B are invertible matrices with the same size, then AB is invertible and* $(AB)^{-1} = B^{-1}A^{-1}$

- The sum of tow invertible matrices needs not to be invertible

$$A=\begin{bmatrix}0 & 1\\ 1 & -3\end{bmatrix} \quad ,\det(A)=0-1=-1\neq 0 \quad \textit{invertible}$$

$$B=\begin{bmatrix}2 & 1\\ 1 & 5\end{bmatrix} \quad ,\det(B)=10-1=9\neq 0 \quad \textit{invertible}$$

$$A+B=\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix} \quad ,\det(A+B)=4-4=0 \quad \textit{not invertible}$$

- The sum of tow not invertible matrices needs not to be not invertible

$$A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \quad ,\det(A)=0-0=0 \quad \textit{not invertible}$$

- $B=\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix} \quad ,\det(B)=0-0=0 \quad \textit{not invertible}$

$$A+B=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \quad ,\det(A+B)=1-0=1 \quad \textit{invertible}$$

EXAMPLE 9 The Inverse of a Product

Consider the matrices

$$A=\begin{bmatrix}1 & 2\\ 1 & 3\end{bmatrix}, \quad B=\begin{bmatrix}3 & 2\\ 2 & 2\end{bmatrix}$$

We leave it for you to show that

$$AB=\begin{bmatrix}7 & 6\\ 9 & 8\end{bmatrix}, \quad (AB)^{-1}=\begin{bmatrix}4 & -3\\ -\frac{9}{2} & \frac{7}{2}\end{bmatrix}$$

and also that

$$A^{-1}=\begin{bmatrix}3 & -2\\ -1 & 1\end{bmatrix}, \quad B^{-1}=\begin{bmatrix}1 & -1\\ -1 & \frac{3}{2}\end{bmatrix}, \quad B^{-1}A^{-1}=\begin{bmatrix}1 & -1\\ -1 & \frac{3}{2}\end{bmatrix}\begin{bmatrix}3 & -2\\ -1 & 1\end{bmatrix}=\begin{bmatrix}4 & -3\\ -\frac{9}{2} & \frac{7}{2}\end{bmatrix}$$

Thus, $(AB)^{-1}=B^{-1}A^{-1}$

Powers of a Matrix If A is a *square* matrix, then we define the nonnegative integer powers of A to be

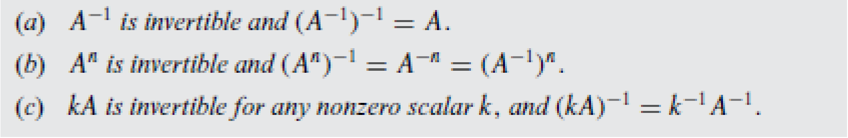
$$A^0=I \quad \text{and} \quad A^n=AA\cdots A \quad \textbf{[}n\textbf{ factors]}$$

and if A is invertible, then we define the negative integer powers of A to be

$$A^{-n}=(A^{-1})^n=A^{-1}A^{-1}\cdots A^{-1} \quad \textbf{[}n\textbf{ factors]}$$

$2^{-3}=(2^3)^{-1}=8^{-1}=1/8$

THEOREM 1.3.7 *If A is invertible and n is a nonnegative integer, then*:



(*a*) A^{-1} *is invertible and* $(A^{-1})^{-1} = A$.

(*b*) A^n *is invertible and* $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.

(*c*) kA *is invertible for any nonzero scalar* k, *and* $(kA)^{-1} = k^{-1}A^{-1}$.

5A

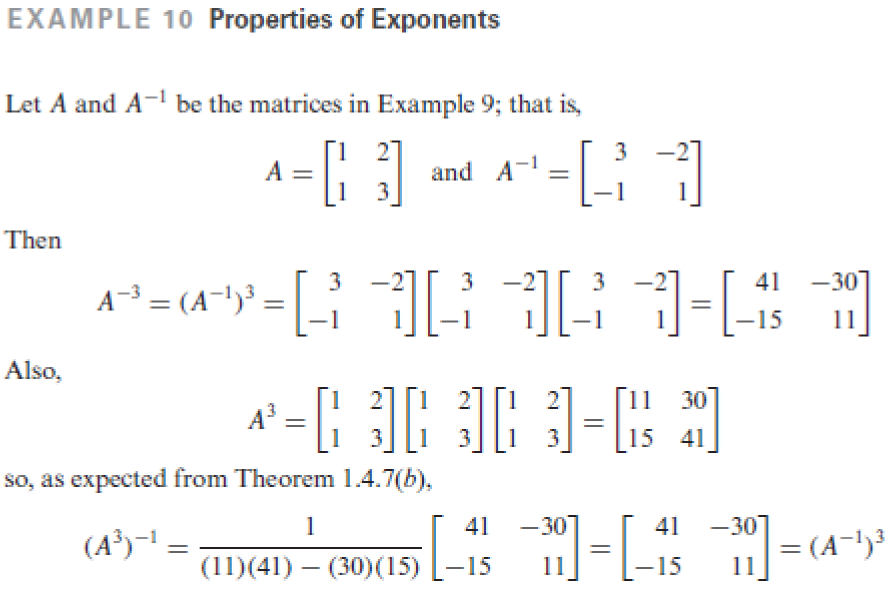
$A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ det(A)=3-0=3

$5A = \begin{bmatrix} 15 & 10 \\ 0 & 5 \end{bmatrix}$, det(5A)=15.5-0.10=75≠0 invertible

$$(5A)^{-1} = \frac{1}{75}\begin{bmatrix} 5 & -10 \\ 0 & 15 \end{bmatrix} = \begin{bmatrix} \frac{1}{15} & \frac{-2}{15} \\ 0 & \frac{1}{5} \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$$

$$5^{-1}A^{-1} = \frac{1}{5}.\frac{1}{3}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{15} & \frac{-2}{15} \\ 0 & \frac{1}{5} \end{bmatrix}$$



EXAMPLE 10 **Properties of Exponents**

Let A and A^{-1} be the matrices in Example 9; that is,

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

Then

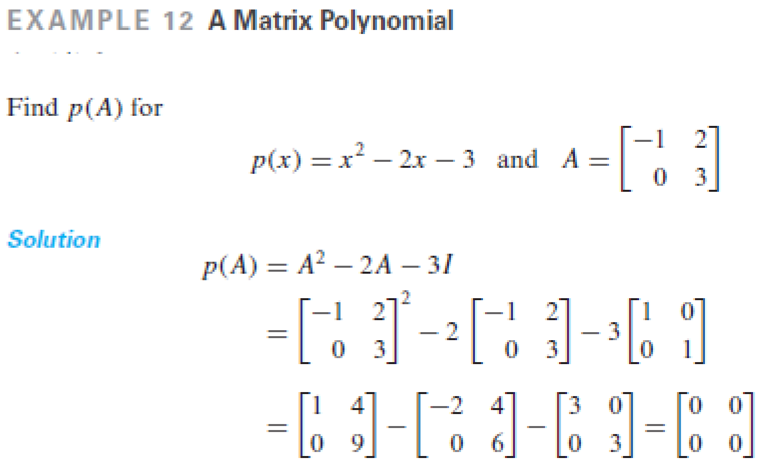
$$A^{-3} = (A^{-1})^3 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

Also,

$$A^3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

so, as expected from Theorem 1.4.7(*b*),

$$(A^3)^{-1} = \frac{1}{(11)(41) - (30)(15)}\begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = (A^{-1})^3$$



EXAMPLE 12 A Matrix Polynomial

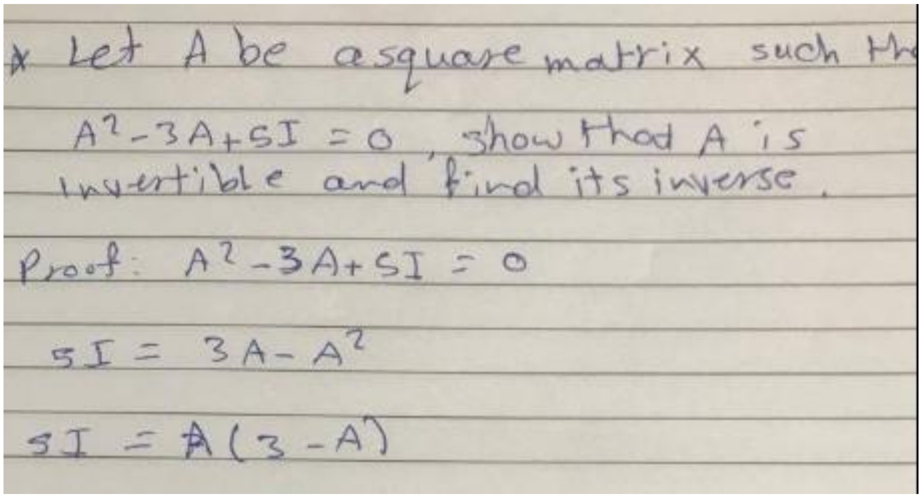
Find $p(A)$ for

$$p(x) = x^2 - 2x - 3 \quad \text{and} \quad A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$$

Solution

$$\begin{aligned} p(A) &= A^2 - 2A - 3I \\ &= \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$A.A^{-1}=A^{-1}.A=I$



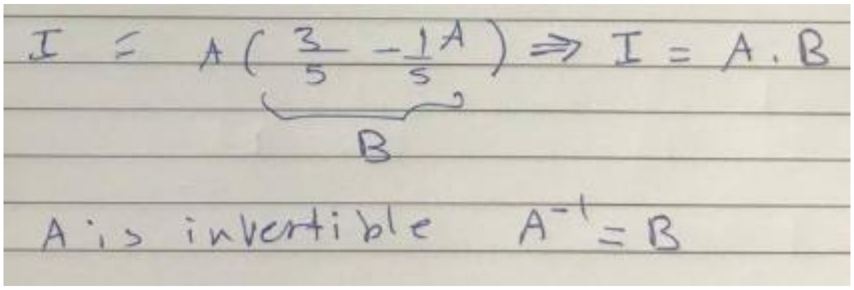
* Let A be a square matrix such th

$A^2-3A+5I=0$, show that A is invertible and find its inverse.

Proof: $A^2-3A+5I=0$

$5I = 3A - A^2$

$5I = A(3-A)$



$I = A(\underbrace{\frac{3}{5} - \frac{1}{5}A}_{B}) \Rightarrow I = A.B$

A is invertible $A^{-1} = B$

- **If** $A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$ compute :

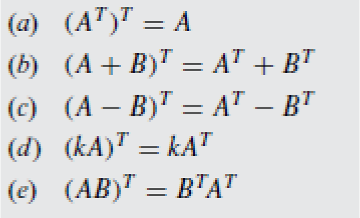
  1- **A^2**

  2- **A^{-3}**

  3- **A^2-2A+I**

  4- **P(x)=x^3-4x^2+7x-18, find P(A)**

THEOREM 1.3.8 *If the sizes of the matrices are such that the stated operations can be performed, then*:



(*a*) $(A^T)^T = A$

(*b*) $(A+B)^T = A^T + B^T$

(*c*) $(A-B)^T = A^T - B^T$

(*d*) $(kA)^T = kA^T$

(*e*) $(AB)^T = B^T A^T$

THEOREM 1.3.9 *If A is an invertible matrix, then A^T is also invertible and*

$(A^T)^{-1} = (A^{-1})^T$

- $A=\begin{bmatrix}2 & 4 & -9\\ 3 & 4 & 2\\ 1 & 7 & 0\end{bmatrix}$, $B=\begin{bmatrix}5 & 0 & 9\\ 2 & 4 & 1\\ 10 & 0 & 0\end{bmatrix}$ compute $(A+B)^T$:

$$A^T=\begin{bmatrix}2 & 3 & 1\\ 4 & 4 & 7\\ -9 & 2 & 0\end{bmatrix},\ B^T=\begin{bmatrix}5 & 2 & 10\\ 0 & 4 & 0\\ 9 & 1 & 0\end{bmatrix}$$

$$A^T+B^T=\begin{bmatrix}7 & 5 & 11\\ 4 & 8 & 7\\ 0 & 3 & 0\end{bmatrix}$$

$$A+B=\begin{bmatrix}7 & 4 & 0\\ 5 & 8 & 3\\ 11 & 7 & 0\end{bmatrix},\ (A+B)^T=\begin{bmatrix}7 & 5 & 11\\ 4 & 8 & 7\\ 0 & 3 & 0\end{bmatrix}$$

Then $A^T+B^T=(A+B)^T$

1.4 Elementary Matrices and a Method for Finding A^{-1}

DEFINITION A matrix *E* is called an ***elementary matrix*** if it can be obtained from an identity matrix by performing a *single* elementary row operation.

Example : Listed below are three elementary matrices and the operations that produce them.

$$E_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2R_1)$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \quad (-3R_1 + R_3)$$

Finding inverse matrix by elementary row operation:

$$\left[A_{n\times n} : I_n\right] \xrightarrow{elemntary\, row\, operation} \left[I_n : A^{-1}{}_{n\times n}\right]$$

EXAMPLE 4 Using Row Operations to Find *A*⁻¹

- Find the inverse of $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ (if it exist)

Solution:

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 2 & -2 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & -1 & \frac{1}{2} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & -1 & \frac{1}{2} \end{array}\right] \xrightarrow{-1R_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right] \xrightarrow{-3R_2 + R_1} \left[\begin{array}{cc|cc} 1 & 0 & -2 & \frac{3}{2} \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right]$$

Then $A^{-1} = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$

- Find the inverse of $A=\begin{bmatrix}6&3\\4&2\end{bmatrix}$(if it exist)

$$\left[\begin{array}{cc|cc}6&3&1&0\\4&2&0&1\end{array}\right]\xrightarrow{\frac{1}{6}R_1}\left[\begin{array}{cc|cc}1&\frac{1}{2}&\frac{1}{6}&0\\4&2&0&1\end{array}\right]$$

$$\left[\begin{array}{cc|cc}1&\frac{1}{2}&\frac{1}{6}&0\\4&2&0&1\end{array}\right]\xrightarrow{-4R_1+R_2}\left[\begin{array}{cc|cc}1&\frac{1}{2}&\frac{1}{6}&0\\0&0&\frac{-4}{6}&1\end{array}\right]$$

A^{-1} does not exist

EXAMPLE 4 Using Row Operations to Find A^{-1}

Find the inverse of $A=\begin{bmatrix}1&2&3\\2&5&3\\1&0&8\end{bmatrix}$

$$\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\2&5&3&0&1&0\\1&0&8&0&0&1\end{array}\right]$$

$-2R_1+R_2$, $-R_1+R_3$

$$\left[\begin{array}{ccc|ccc}1&2&3&1&0&0\\0&1&-3&-2&1&0\\0&-2&5&-1&0&1\end{array}\right]$$

$2R_2+R_3$,

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

-R$_3$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

3R$_3$+R$_2$, -3R$_3$+R$_1$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

-2R$_2$+R$_1$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

EXAMPLE Showing That a Matrix Is Not Invertible

Consider the matrix $A=\begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$-2R_1+R_2$, R_1+R_3

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$$

R_2+R_3

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

Since we have obtained a row of zeros on the left side, A is not invertible.

THEOREM *If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix $\mathbf{b}$, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely, $\mathbf{x} = A^{-1}\mathbf{b}$.*

$A^{-1}(Ax=b)$, $A^{-1}Ax=A^{-1}b$, $Ix=A^{-1}b$, $x=A^{-1}b$

Example: Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 5 \\ 2x_1 + 5x_2 + 3x_3 &= 3 \\ x_1 \qquad\quad + 8x_3 &= 17 \end{aligned}$$

In matrix form this system can be written as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

A is invertible and

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

the solution of the system is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

or $x_1 = 1$, $x_2 = -1$, $x_3 = 2$.

Example: Solve the systems

(a) $$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 5x_2 + 3x_3 &= 5 \\ x_1 \qquad\quad + 8x_3 &= 9 \end{aligned}$$

(b) $$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 3x_3 &= 6 \\ x_1 \qquad\quad + 8x_3 &= -6 \end{aligned}$$

Solution The two systems have the same coefficient matrix. If we augment this coefficient matrix with the columns of constants on the right sides of these systems, we obtain

$$\left[\begin{array}{ccc|c|c} 1 & 2 & 3 & 4 & 1 \\ 2 & 5 & 3 & 5 & 6 \\ 1 & 0 & 8 & 9 & -6 \end{array}\right]$$

Reducing this matrix to reduced row echelon form yields (verify)

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

Diagonal, Triangular, and Symmetric Matrices

Diagonal Matrices: A square matrix in which all the entries off the main diagonal are zero is called a ***diagonal matrix***.

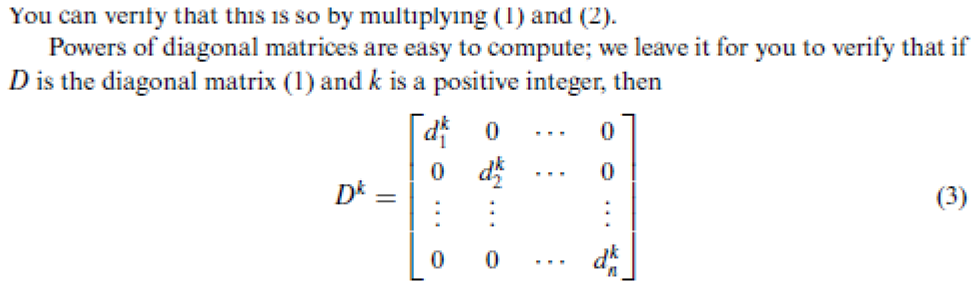
$$A=\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A general $n \times n$ diagonal matrix D can be written as

$$D=\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \tag{1}$$

A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero; in this case the inverse of (1) is

$$D^{-1}=\begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix} \tag{2}$$



You can verify that this is so by multiplying (1) and (2).

Powers of diagonal matrices are easy to compute; we leave it for you to verify that if D is the diagonal matrix (1) and k is a positive integer, then

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix} \tag{3}$$

Example: if $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **, find A^{-1}, A^{3}, A^{-3}, A^{2}**

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{3} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-3} = \begin{bmatrix} \frac{1}{8} & 0 & 0 \\ 0 & \frac{1}{27} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{2} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Triangular Matrices A square matrix in which all the entries above the main diagonal are zero is called ***lower triangular***, and a square matrix in which all the entries below the main diagonal are zero is called ***upper triangular***. A matrix that is either upper triangular or lower triangular is called ***triangular***.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -1 & 9 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 9 & 1 \\ 0 & 9 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

THEOREM

(*a*) *The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.*

(*b*) *The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.*

(*c*) *A triangular matrix is invertible if and only if its diagonal entries are all nonzero.*

(*d*) *The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular*

Example:

Consider the upper triangular matrices

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

the matrix *A* is invertible but the matrix *B* is not. Moreover, the theorem also tells us that A^{-1}, *AB*, and *BA* must be upper triangular.

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}, \quad AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}, \quad BA = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

DEFINITION: A square matrix *A* is said to be ***symmetric*** if $A = A^T$.

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 3 & 5 & 5 \\ 0 & -2 & 7 \end{bmatrix} \quad A^T = \begin{bmatrix} 3 & 3 & 0 \\ -3 & 5 & -2 \\ 4 & 5 & 7 \end{bmatrix}$$

Remark: 1- If A is diagonal then $A = A^T$

2-Every diagonal matrix is symmetric

Example: $A=\begin{bmatrix}3 & 0\\ 0 & -1\end{bmatrix},\ A^T=\begin{bmatrix}3 & 0\\ 0 & -1\end{bmatrix}=A$

THEOREM *If A and B are symmetric matrices with the same size, and if k is any scalar, then*:

(a) A^T *is symmetric.*

Proof: since A is symmetric then $A^T=A \rightarrow (A^T)^T = A^T \rightarrow A=A^T$

(*b*) *A + B and A − B are symmetric.*

(*c*) *kA is symmetric.*

(d) AB is symmetric if and only if AB=BA

$(A.B)^T= B^T.A^T = B.A=AB$

Conversely

If $(A.B)^T=A.B$ and $(A.B)^T= B^T.A^T = B.A$

A.B=B.A

THEOREM *If A is an invertible symmetric matrix, then* A^{-1} *is symmetric.*

THEOREM *If A is an invertible matrix, then* AA^T *and* A^TA *are also invertible.*

Example: Find all value of a,b and c for which A is symmetric

$$A=\begin{bmatrix}3 & a-2b-2c & 2a+b+c\\ 3 & 5 & a+c\\ 0 & -2 & 7\end{bmatrix}$$

Solution: $A=A^T$

$$\begin{bmatrix}3 & a-2b-2c & 2a+b+c\\ 3 & 5 & a+c\\ 0 & -2 & 7\end{bmatrix}=\begin{bmatrix}3 & 3 & 0\\ a-2b-2c & 5 & -2\\ 2a+b+c & a+c & 7\end{bmatrix}$$

$\rightarrow$ a-2b-2c = 3

2a+b+c = 0

a+c = -2

a+c= -2 $\rightarrow$ c = -2-a

a-2b-2c = 3 $\rightarrow$ a-2b-2(-2-a)= 3

a-2b+4+2a = 3

3a-2b = -1

2a+b+c = 0 $\rightarrow$ 2a+b-2-a=0

a+b= 2

3a-2b = -1 --------- (1)

a+b= 2 ----------(2)

2 eq(2) +eq (1)$\rightarrow$ 5a = 3 $\rightarrow$a = 3/5 , b= 7/5 and c= -13/5

Definition: A square matrix *A* is said to be ***skew-symmetric*** if $A^T = -A$.

THEOREM *If A and B are skew- symmetric matrices with the same size, and if k is any scalar, then*:

(a) A^T *is skew-symmetric.*

(*b*) *A + B and A − B are skew-symmetric.*

(*c*) *kA is skew- symmetric.*

THEOREM *If A is an invertible skew-symmetric matrix, then* A^{-1} *is skew- symmetric.*

The procedure (or algorithm) we have just described for reducing a matrix to reduced row echelon form is called ***Gauss–Jordan elimination***. This algorithm consists of two parts, a ***forward phase*** in which zeros are introduced below the leading 1’s and a ***backward phase*** in which zeros are introduced above the leading 1’s. If only the forward phase is used, then the procedure produces a row echelon form and is called ***Gaussian elimination***.

EXAMPLE: Gauss–Jordan Elimination

Solve by Gauss–Jordan elimination.

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 + 2x_5 &= 0\\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1\\
5x_3 + 10x_4 + 15x_6 &= 5\\
2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 6
\end{aligned}$$

Solution The augmented matrix for the system is

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 2 & 6 & -5 & -2 & 4 & -3 & -1\\ 0 & 0 & 5 & 10 & 0 & 15 & 5\\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

$-2R_1+R_2$, $-2R_1+R_4$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 0 & 0 & -1 & -2 & 0 & -3 & -1\\ 0 & 0 & 5 & 10 & 0 & 15 & 5\\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}$$

$-R_2$ then $-5R_2+R_3$, $-4R_2+R_4$, $2R_2+R_1$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 2 & 0 & 3 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}$$

Interchanging the third and fourth rows and then multiplying the third row of the resulting matrix by $\frac{1}{6}$ gives the row echelon form

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 2 & 0 & 3 & 1\\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3}\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$-3R_3+R_2$, $2R_2+R_1$

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0\\ 0 & 0 & 1 & 2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3}\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 \quad + 4x_4 + 2x_5 \quad &= 0 \\ x_3 + 2x_4 \quad &= 0 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

Solving for the leading variables, we obtain

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

Finally, we express the general solution of the system parametrically by assigning the free variables x_2, x_4, and x_5 arbitrary values r, s, and t, respectively. This yields

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = \tfrac{1}{3}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r-4s-2t \\ r \\ -2s \\ s \\ t \\ 1\backslash 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1\backslash 3 \end{bmatrix} + r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

r=1 , s=2 , t=0

x_1=-11 , x_2=1 , x_3=-4, x_4=2, x_5=0 , x_6=1\3

Homogeneous Linear Systems

A system of linear equations is said to be ***homogeneous*** if the constant terms are all zero; that is, the system has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\quad \vdots \quad &\ \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Every homogeneous system of linear equations is consistent because all such systems have $x_1 = 0, x_2 = 0, . . . , x_n = 0$ as a solution. This solution is called the ***trivial solution***; if there are other solutions, they are called ***nontrivial solutions***.

Because a homogeneous linear system always has the trivial solution, there are only two possibilities for its solutions:

- The system has only the trivial solution.
- The system has infinitely many solutions in addition to the trivial solution.

EXAMPLE: A Homogeneous System

Use Gauss–Jordan elimination to solve the homogeneous linear system

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 + 15x_6 &= 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

Solution

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}$$

which is the same as the augmented matrix for the system in previous Example , except for zeros in the last column. Thus, the reduced row echelon form of this matrix will be the same as that of the augmented matrix in Example 5, except for the last column. However, a moment's reflection will make it evident that a column of zeros is not changed by an elementary row operation, so the reduced row echelon is

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 + 4x_4 + 2x_5 &= 0 \\ x_3 + 2x_4 &= 0 \\ x_6 &= 0 \end{aligned}$$

Solving for the leading variables, we obtain

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= 0 \end{aligned}$$

If we now assign the free variables x_2, x_4, and x_5 arbitrary values r, s, and t, respectively, then we can express the solution set parametrically as

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r-4s-2t \\ r \\ -2s \\ s \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

r=1,s=0,t=0

Note that the trivial solution results when $r = s = t = 0$.

CHAPTER 2

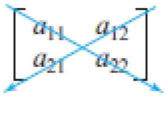
Determinants

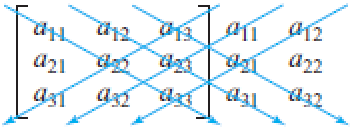
2.1 Determinants by Cofactor Expansion

- A= $[0] \rightarrow \quad \det(A) = 0$
- A= $[a_{11}] \rightarrow \quad \det(A) = a_{11}$
- A= $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \quad \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a.d - c.b$
- A= $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow \quad \det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$$\det(A) = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Determinants of 2 × 2 and 3 × 3 matrices can be evaluated very efficiently using the pattern suggested in Figure





det=$a_{11}.a_{22}-a_{12}.a_{21}$

EXAMPLE: Find the determinant of the matrix

1) A= $\begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$

$$\det(A) = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3\,(8-12) - 1\,(4-15) + 0$$

$$= -12 + 11 + 0 = -1$$

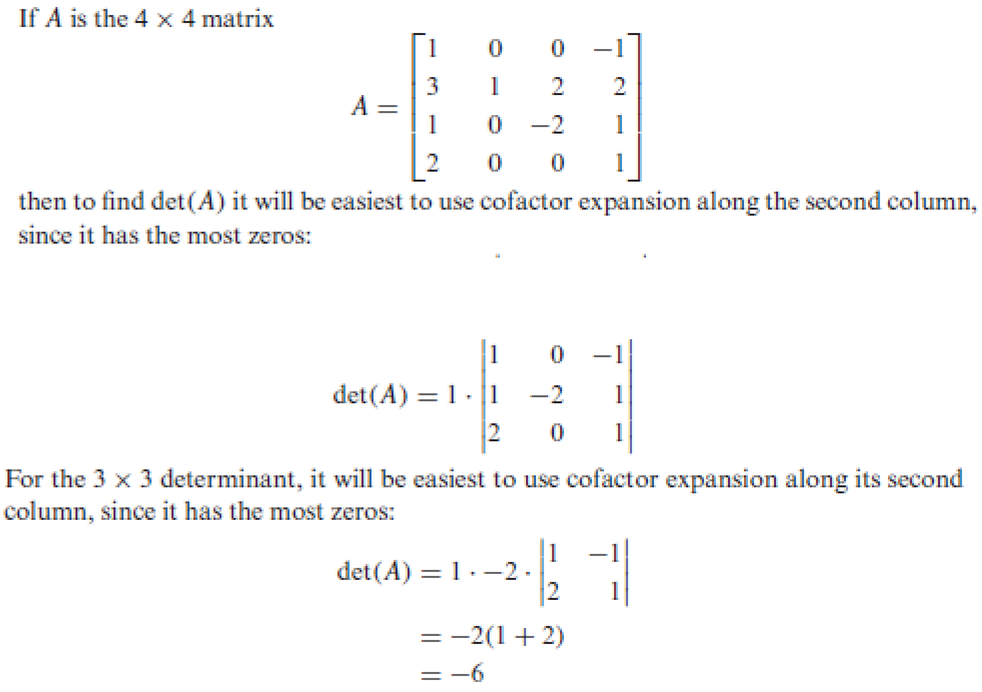
OR

$$\det(A) = \left|\begin{matrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{matrix}\right|\begin{matrix} 3 & 1 \\ -2 & -4 \\ 5 & 4 \end{matrix}$$

$$= (24 + 15 + 0) - (0 + 36 + 4)$$

$$= 39 - 40 = -1$$

EXAMPLE:



If A is the 4 × 4 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

then to find $\det(A)$ it will be easiest to use cofactor expansion along the second column, since it has the most zeros:

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

For the 3 × 3 determinant, it will be easiest to use cofactor expansion along its second column, since it has the most zeros:

$$\begin{aligned} \det(A) &= 1 \cdot -2 \cdot \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \\ &= -2(1+2) \\ &= -6 \end{aligned}$$

2) A= $\begin{bmatrix} 1 & 2 & -3 & 1 \\ 2 & 3 & 0 & 1 \\ 2 & 1 & -1 & 2 \\ 3 & 0 & 0 & 4 \end{bmatrix}$

THEOREM: If A is an n × n triangular matrix (upper triangular, lower triangular, or diagonal), then det(A) is the product of the entries on the main diagonal of the matrix; that is, det(A) = $a_{11}\, a_{22} \cdots a_{nn}$.

EXAMPLE: evaluate the determinant of the given matrix

$$\begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

det = 1.1.2.3= 6

2- Evaluating Determinants by Row Reduction

THEOREM: Let A be a square matrix. If A has a row of zeros or a column of zeros, then det(A) = 0.

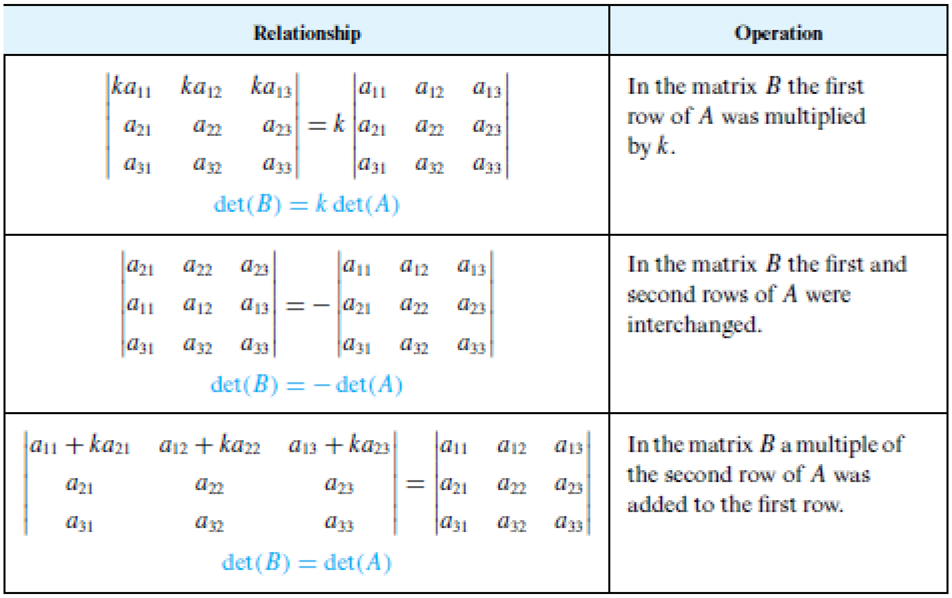
THEOREM: Let A be a square matrix. Then det(A) = det(A^T).

THEOREM: Let A be an n × n matrix.

(a) If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then det(B) = k det(A).

(b) If B is the matrix that results when two rows or two columns of A are interchanged, then det(B) = −det(A).

(c) If B is the matrix that results when a multiple of one row of A is added to another or when a multiple of one column is added to another, then det(B) = det(A).



Relationship	Operation
$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = k\det(A)$	In the matrix B the first row of A was multiplied by k.
$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = -\det(A)$	In the matrix B the first and second rows of A were interchanged.
$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = \det(A)$	In the matrix B a multiple of the second row of A was added to the first row.

EXAMPLE:

$$A = \begin{bmatrix} 3 & 5 \\ 2 & 1 \end{bmatrix} \rightarrow \det(A) = -7$$

$$B = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix} \rightarrow \det(B) = -14 = 2.\det(A)$$

$$C = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \rightarrow \det(C) = 7 = -\det(A)$$

THEOREM: Let E be an n × n elementary matrix.

(a) If E results from multiplying a row of In by a nonzero number k, then det(E) = k.

(b) If E results from interchanging two rows of In, then det(E) = −1.

(c) If E results from adding a multiple of one row of In to another, then det(E) = 1.

EXAMPLE:

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 3, \qquad \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -1$$

The second row of I_4 was multiplied by 3.

The first and last rows of I_4 were interchanged.

$$\begin{vmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

7 times the last row of I_4 was added to the first row.

THEOREM: If A is a square matrix with two proportional rows or two proportional columns, then det(A) = 0.

EXAMPLE:

$$A = \begin{bmatrix} 3 & -12 \\ 2 & -8 \end{bmatrix} \rightarrow \det(A) = 0 \quad (C_2 = -4C_1)$$

$$B = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 5 & 0 \\ 6 & 2 & 4 \end{bmatrix} \rightarrow \det(B) = 0 \quad (R_3 = 2.R_1)$$

EXAMPLE:

Evaluate $\det(A)$ where

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

← The first and second rows of A were interchanged.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

← A common factor of 3 from the first row was taken through the determinant sign.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

← -2 times the first row was added to the third row.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

-10 times the second row was added to the third row.

$$= (-3)(-55)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$

A common factor of -55 from the last row was taken through the determinant sign.

$$= (-3)(-55)(1) = 165$$

EXAMPLE:

Evaluate $\det(A)$ where

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

Solution By adding suitable multiples of the second row to the remaining rows, we obtain

$$\det(A) = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

Cofactor expansion along the first column

$$= -\begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix}$$

We added the first row to the third row.

$$= -(-1)\begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix}$$

Cofactor expansion along the first column

$$= -18$$ ◀

EXAMPLE:

If $\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -6$, find

1- $\det\begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix} = \text{-}6$

2- $\det\begin{bmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{bmatrix} = 3.-1.4.-6 = 72$

3- $\det\begin{bmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{bmatrix} = -6$

4- $\det\begin{bmatrix} 3a & 3b & 3c \\ d & e & f \\ g-3d & h-3e & i-3f \end{bmatrix} = 3.-6 = -18$

EXAMPLE:

Evaluate the determinant of the matrix by first reducing the matrix to row echelon form and then using some combination of row operations and cofactor expansion.

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} (R_1 \leftrightarrow R_2)$$

$$= -\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} (-2R_1 + R_2)$$

$$= -\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 4 \end{vmatrix} (-2R_2 + R_3), (-R_2 + R_4)$$

$$= \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 4 \end{vmatrix} (-R_3)$$

$$= \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 6 \end{vmatrix} (-R_3 + R_4)$$

=1.1.1.6 = 6

2.3 Properties of Determinants; Cramer's Rule

Suppose that *A* and *B* are *n* × *n* matrices and *k* is any scalar.

1. det(kA) = k^n det(A)

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} = k^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

2. det(*A* + *B*) ≠ det(*A*) + det(*B*)

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad A + B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$$

We have $\det(A) = 1$, $\det(B) = 8$, and $\det(A + B) = 23$; thus

$$\det(A + B) \neq \det(A) + \det(B)$$

3. det(AB) = det(A) det(B)

Consider the matrices

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix}, \quad AB = \begin{bmatrix} 2 & 17 \\ 3 & 14 \end{bmatrix}$$

We leave it for you to verify that

$$\det(A) = 1, \quad \det(B) = -23, \quad \text{and} \quad \det(AB) = -23$$

Thus $\det(AB) = \det(A)\det(B)$

THEOREM Let A, B, and C be n × n matrices that differ only in a single row, say the rth, and assume that the rth row of C can be obtained by adding corresponding entries in the rth rows of A and B. Then

det(C) = det(A) + det(B)

The same result holds for columns.

Example:

$$\det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7+(-1) \end{bmatrix} = \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{bmatrix} + \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\det\begin{bmatrix} 4 & 5 & -2 \\ 3 & 0 & 7 \\ 9 & 10 & 4 \end{bmatrix} = \det\begin{bmatrix} 4 & 5 & -2 \\ 3 & 0 & 7 \\ 4 & 5 & 3 \end{bmatrix} + \det\begin{bmatrix} 4 & 5 & -2 \\ 3 & 0 & 7 \\ 5 & 5 & 1 \end{bmatrix}$$

If $A = \begin{bmatrix} 4 & 5 & -2 \\ 3 & 0 & 7 \\ 4 & 5 & 3 \end{bmatrix}$**,** $B = \begin{bmatrix} 4 & 5 & -2 \\ 3 & 0 & 7 \\ 5 & 5 & 1 \end{bmatrix}$ **Find det(A)+det(B)**

THEOREM *If A is invertible, then*

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Example:

Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, and assume that det(A)= -5

Find:

1. det(3A)= 3^3 det(A)= 27. -5= -135
2. $\det(A^{-1}) = \dfrac{1}{\det(A)} = \dfrac{1}{-5}$
3. $\det(2A^{-1}) = 2^3 \det(A^{-1}) = 8\dfrac{1}{\det(A)} = -\dfrac{8}{5}$
4. $\det(2A)^{-1} = \dfrac{1}{\det(2A)} = \dfrac{1}{2^3 \det(A)} = \dfrac{1}{8.-5} = -\dfrac{1}{40}$

Cofactor expansion

Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

$$C_{11} = +\begin{vmatrix} 6 & 3 \\ -4 & 0 \end{vmatrix} = 0 + 12 = 12$$

$$C_{12} = -\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = -(0-6) = 6$$

$$C_{13} = +\begin{vmatrix} 1 & 6 \\ 2 & -4 \end{vmatrix} = -4 - 12 = -16 \,.......$$

$C_{11} = 12$ $\quad C_{12} = 6$ $\quad C_{13} = -16$
$C_{21} = 4$ $\quad C_{22} = 2$ $\quad C_{23} = 16$
$C_{31} = 12$ $\quad C_{32} = -10$ $\quad C_{33} = 16$

$$C = \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

DEFINITION 1 If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the ***matrix of cofactors from A***. The transpose of this matrix is called the ***adjoint of A*** and is denoted by adj(A).

Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

Find adj(A)

$$\text{Adj(A)}=C^T=\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

THEOREM Inverse of a Matrix Using Its Adjoint

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

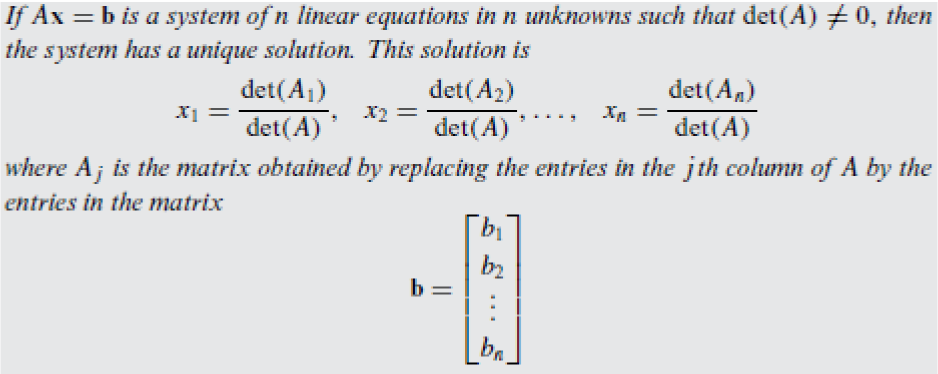
Let

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

Find A^{-1}

$$A^{-1} = \frac{1}{\det(A)} adj(A) = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

THEOREM Cramer's Rule



If $A\mathbf{x} = \mathbf{b}$ *is a system of n linear equations in n unknowns such that* $\det(A) \neq 0$, *then the system has a unique solution. This solution is*

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j *is the matrix obtained by replacing the entries in the* j*th column of A by the entries in the matrix*

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Example:

Use Cramer's rule to solve

$$\begin{aligned} x_1 + \quad\;\; + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

Solution

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

Therefore,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11},$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11} \quad \blacktriangleleft$$

2)

$$\begin{aligned} -x_1 - 4x_2 + 2x_3 + \;\; x_4 &= -32 \\ 2x_1 - \;\; x_2 + 7x_3 + 9x_4 &= 14 \\ -x_1 + \;\; x_2 + 3x_3 + \;\; x_4 &= 11 \\ x_1 - 2x_2 + \;\; x_3 - 4x_4 &= -4 \end{aligned}$$

3)

$$\begin{aligned} x_1 - 3x_2 + \;\; x_3 &= 4 \\ 2x_1 - \;\; x_2 \qquad\;\; &= -2 \\ 4x_1 \qquad\quad - 3x_3 &= 0 \end{aligned}$$

CHAPTER 3

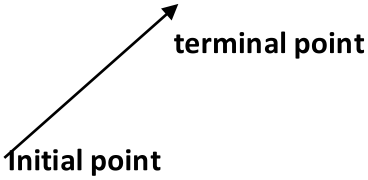
Euclidean Vector Spaces

3.1Vectors in 2-Space, 3-Space, and n-Space

Geometric Vectors

Engineers and physicists represent vectors in two dimensions (also called **2-space**) or in three dimensions (also called **3-space**) by arrows. The direction of the arrowhead specifies the **direction** of the vector and the **length** of the arrow specifies the magnitude.

Mathematicians call these **geometric vectors**. The tail of the arrow is called the **initial point** of the vector and the tip the **terminal point.**



3 Euclidean Vector Spaces

3.1 Euclidean n-space

In this chapter we will generalize the findings from last chapters for a space with n dimensions, called n-space.

$R^2=\{(x,y): x,y \in R\}$

$R^4=\{(x,y,z,l): x,y,z,l \in R\}$

Definition 1

If n $\in N \backslash \{0\}$ then an ordered n-tuple is a sequence of n numbers in R: $(a_1, a_2, . . . , a_n)$.

The set of all ordered n-tuples is called n-space and is denoted by R^n.

The elements in R^n can be perceived as points or vectors, similar to what we have done in 2- and 3-space. (a_1, a_2, a_3) was used to indicate the components of a vector or the coordinates of a point.

Example: $R^2 = \{(x,y): x,y \in R\}$

$R^3 =\{(x,y,z): x,y,z \in R\}$

Definition 2

Two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n are called equal if

$u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n$

The sum $\mathbf{u} + \mathbf{v}$ is defined by

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots u_n + v_n)$$

If $k \in R$ the scalar multiple of $\mathbf{u}$ is defined by

$k\mathbf{u} = (ku_1, ku_2, \ldots, ku_n)$ These

operations are called the standard operations in R^n.

Definition 3

The zero vector $\mathbf{0}$ in R^n is defined by

$$\mathbf{0} = (0, 0, \ldots, 0)$$

For $\mathbf{u} = (u_1, u_2, \ldots, u_n) \in R^n$ the negative of $\mathbf{u}$ is defined by

$-\mathbf{u} = (-u_1, -u_2, \ldots, -u_n)$

The difference between two vectors $\mathbf{u}, \mathbf{v} \in R^n$ is defined by

$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$

Theorem 1

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in R^n and $k, l \in R$, then

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(c) $\mathbf{u} + \mathbf{0} = \mathbf{u}$

(d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(e) $k(l\mathbf{u}) = (kl)\mathbf{u}$

(f) $k(\mathbf{u}+\mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(g) $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$

(h) $1\mathbf{u} = \mathbf{u}$

This theorem permits us to manipulate equations without writing them in component form.

Definition 4

If $\mathbf{u} = (u_1, u_2, \ldots, u_n), \mathbf{v} = (v_1, v_2, \ldots, v_n) \in R^n$, then the Euclidean inner product $\mathbf{u}\cdot\mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \ldots u_nv_n$$

Example u=(3,4,-1,2) , v=(0,-1,5,6)

u.v =0+-4+-5+12= 3

Theorem 2

If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in R^n and $k \in R$, then

(a) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

(b) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$

(c) $(k\mathbf{u})\cdot\mathbf{v} = k(\mathbf{u}\cdot\mathbf{v})$

(d) $\mathbf{u}\cdot\mathbf{u} \geq 0$.

(e) $\mathbf{u}\cdot\mathbf{u} = 0$ if and only if $\mathbf{u} = 0$.

Proof:

(d) Let $\mathbf{u} \in R^n$ then $\mathbf{u} \cdot \mathbf{u} = u_1^2+u^1$, by definition. Since all terms are squares they are greater or equal than zero, and since the sum of numbers greater or equal than zero is also greater or equal than zero we found that $\mathbf{u} \cdot \mathbf{u} \geq 0$.

The total can only be zero if each individual term is zero, that is $u^2_i = 0$ for all $1 \leq i \leq n$, but this is equivalent to $u_i = 0$ for $1 \leq i \leq n$, therefore $\mathbf{u} = \mathbf{0}$, which proves (e).

Definition 5

If $\mathbf{u} \in R^n$ then the Euclidean norm of $\mathbf{u}$ is defined by

$$||\mathbf{u}|| = \sqrt{u.u}$$

The Euclidean distance between two points $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{v} - \mathbf{u}||$$

Theorem 3

Cauchy-SchwarzInequalityinR^n

If $\mathbf{u}, \mathbf{v} \in R^n$, then

$$|u.v| \leq \|u\| \|v\|$$

Theorem 4

If $\mathbf{u}, \mathbf{v} \in R^n$, and $k \in R$, then:

(a) $||\mathbf{u}|| \geq 0$

(b) $||\mathbf{u}|| = 0$ if and only if $\mathbf{u} = \mathbf{0}$

(c) $||k\mathbf{u}|| = |k| \, ||\mathbf{u}||$

(d) $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$ (triangle inequality)

Proof

(d) Let **u**, **v** $\in$ R^n, then

$||\mathbf{u}+\mathbf{v}||^2 = (\sqrt{(u+v).(u+v)})^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$

$= \mathbf{u}\cdot\mathbf{u} + 2\,\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}$

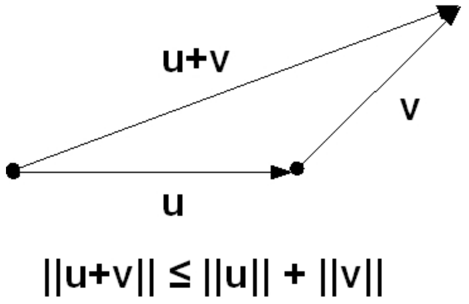
$= ||\mathbf{u}||^2 + 2\,\mathbf{u}\cdot\mathbf{v} + ||\mathbf{v}||^2$

$\leq ||\mathbf{u}||^2 + 2|\mathbf{u}\cdot\mathbf{v}| + ||\mathbf{v}||^2$ absolute value

$\leq ||\mathbf{u}||^2 + 2||\mathbf{u}||\,||\mathbf{v}|| + ||\mathbf{v}||^2$ Cauchy – Schwarz

$= (||\mathbf{u}|| + ||\mathbf{v}||)^2$

Then the triangle inequality follows by taking the square root on both sides.



Theorem 5

If **u**, **v** and **w** in R^n and k ∈ R, then:

(a) d(**u**, **v**) ≥ 0

(b) d(**u**, **v**) = 0 if and only if **u** = **v**

(c) d(**u**, **v**) = d(**v**,**u**)

(d) d(**u**, **v**)≤ d(**u**, **w**) + d(**w**, **v**) Triangle inequality

Theorem 6

If **u**, **v** $\in$ R^n, then:

$$u\,.\,v = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2$$

Definition 6

If **u**, **v** $\in$ R^n, then:

Two vectors **u**, **v** $\in$ R^n are called orthogonal if **u** $\cdot$ **v** = 0.

Example : Let u=(-2, 3, 1 , 4), v=(1, 2 ,0 ,-1) show that **u**, **v** are orthogonal.

u . v = -2+6+0+-4 = 0

then $u \perp v$ (orthogonal)

Motivated by a result in R^2 and R^3 we find

Theorem 7

If **u** and **v** are orthogonal in R^n, then

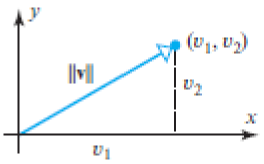
$$||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$$

Proof: Let **u**, **v** be orthogonal vectors in R^n , then **u** $\cdot$ **v** = 0, therefore

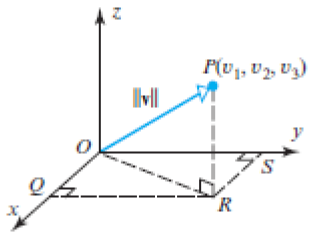
$||\mathbf{u}+\mathbf{v}||^2$ = **(u + v) (u + v)=** $||\mathbf{u}||^2$ **+ 2 u . v +** $||\mathbf{v}||^2$

= $||\mathbf{u}||^2 + ||\mathbf{v}||^2$

Norm of a Vector



(a)



(b)

Unit Vectors: A vector of norm 1 is called a **unit vector**.

More generally, if **v** is any non zero vector in R^n, then

$$u = \frac{1}{\|v\|} v$$

EXAMPLE

Find the unit vector $\mathbf{u}$ that has the same direction as $\mathbf{v} = (2, 2, -1)$.

Solution The vector $\mathbf{v}$ has length

$$\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$$

Thus, from (4)

$$\mathbf{u} = \tfrac{1}{3}(2, 2, -1) = \left(\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{1}{3}\right)$$

$$\|u\| = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = \sqrt{1} = 1$$

EXAMPLE

If

$$\mathbf{u} = (1, 3, -2, 7) \quad \text{and} \quad \mathbf{v} = (0, 7, 2, 2)$$

then the distance between $\mathbf{u}$ and $\mathbf{v}$ is

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(1-0)^2 + (3-7)^2 + (-2-2)^2 + (7-2)^2} = \sqrt{58}$$

The dot product and matrix multiplication

Vectors in R^n can be interpreted as 1 × n or n × 1 matrices. We will identify vectors in R^n with column vectors in matrix notation, that is n × 1 matrices.

In this case the scalar multiplication and addition in the Euclidean space is equivalent to the scalar multiplication and addition for matrices, respectively.

$$u = \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ u_n \end{bmatrix} \text{ or } u = \begin{bmatrix} u_1 & u_2 & . & . & u_n \end{bmatrix}$$

$$u+v=\begin{bmatrix}u_1\\u_2\\.\\.\\u_n\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\.\\.\\v_n\end{bmatrix}=\begin{bmatrix}u_1+v_1\\u_2+v_2\\.\\.\\u_n+v_n\end{bmatrix}$$

$$ku=\begin{bmatrix}ku_1\\ku_2\\.\\.\\ku_n\end{bmatrix}$$

Or

$$u+v=\begin{bmatrix}u_1 & u_2 & . & . & u_n\end{bmatrix}+\begin{bmatrix}v_1 & v_2 & . & , & v_n\end{bmatrix}=\begin{bmatrix}u_1+v_1 & u_2+v_2 & . & . & u_n+v_n\end{bmatrix}$$

For the dot product and the matrix multiplication of two vectors **u, v** $\in R^n$ the following relationship holds:

u · v = u^T v = v^T u

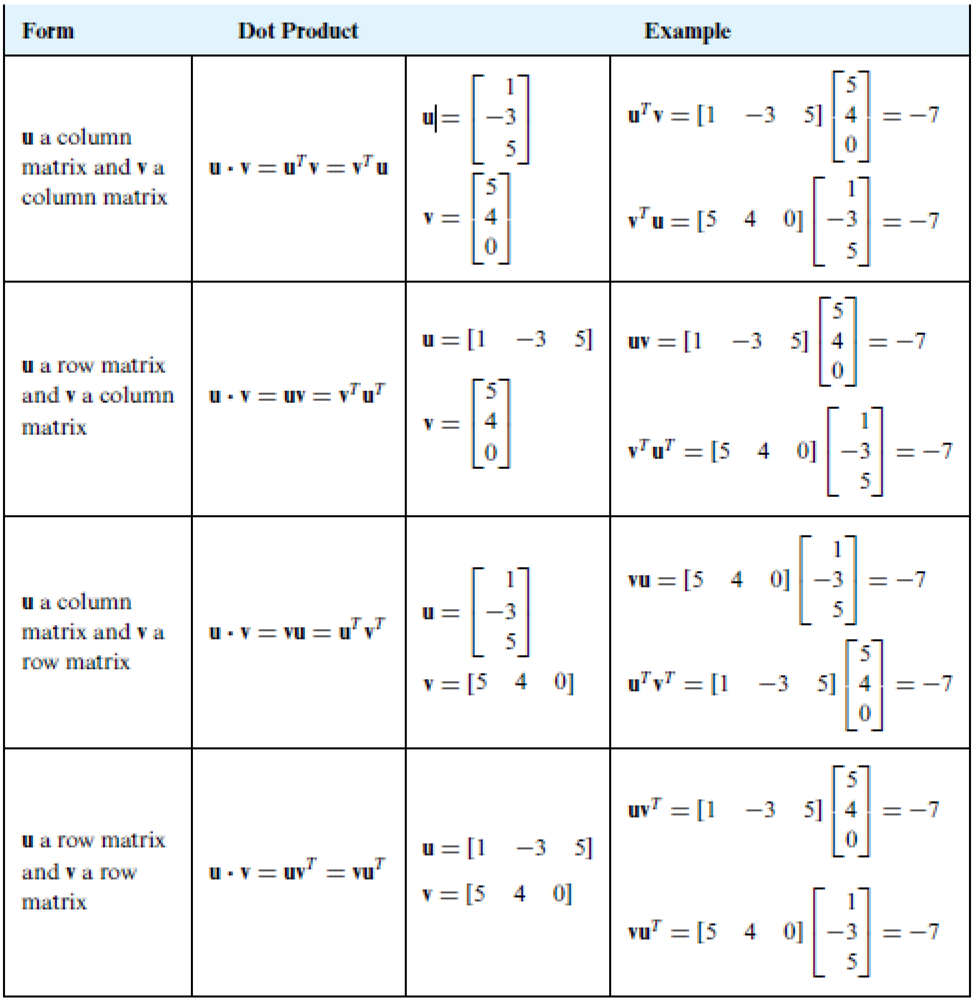
$$u\,.\,v=\begin{bmatrix}u_1 & u_2 & . & . & u_n\end{bmatrix}\begin{bmatrix}v_1\\v_2\\.\\.\\v_n\end{bmatrix}=u_1v_1+u_2v_2+.......+u_nv_n$$

Example: $u=\begin{bmatrix}5\\-4\\7\\0\end{bmatrix},\quad v=\begin{bmatrix}-1\\3\\6\\5\end{bmatrix}$

$$u\,.\,v=\begin{bmatrix}5 & -4 & 7 & 0\end{bmatrix}\begin{bmatrix}-1\\3\\6\\5\end{bmatrix}=-5+-12+42+0=25$$

Or

$$u\,.\,v = \begin{bmatrix} -1 & 3 & 6 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ -4 \\ 7 \\ 0 \end{bmatrix} = \quad -5 + -12 + 42 + 0 = 25$$



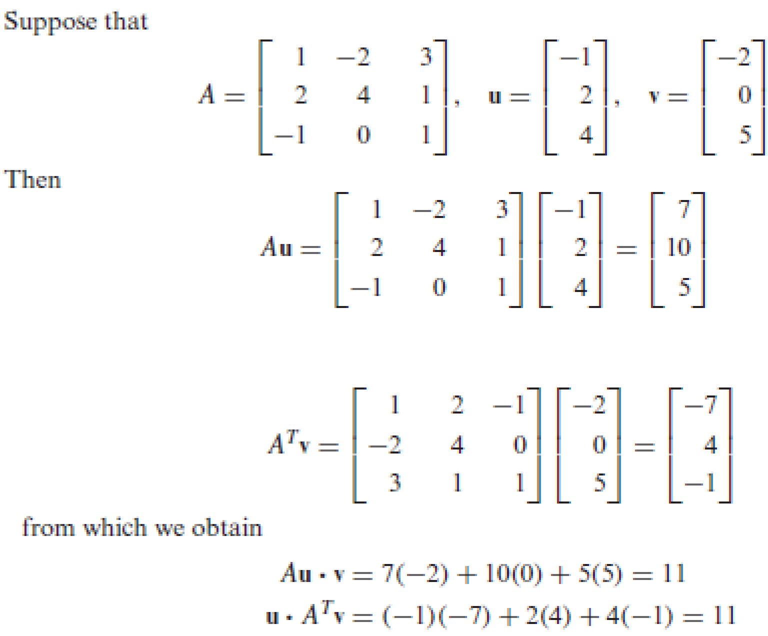
Form	Dot Product	Example	
u a column matrix and **v** a column matrix	$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}^T\mathbf{v} = \mathbf{v}^T\mathbf{u}$	$\mathbf{u} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$ $\mathbf{v} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$	$\mathbf{u}^T\mathbf{v} = [1 \quad -3 \quad 5]\begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$ $\mathbf{v}^T\mathbf{u} = [5 \quad 4 \quad 0]\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$
u a row matrix and **v** a column matrix	$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\mathbf{v} = \mathbf{v}^T\mathbf{u}^T$	$\mathbf{u} = [1 \quad -3 \quad 5]$ $\mathbf{v} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$	$\mathbf{u}\mathbf{v} = [1 \quad -3 \quad 5]\begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$ $\mathbf{v}^T\mathbf{u}^T = [5 \quad 4 \quad 0]\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$
u a column matrix and **v** a row matrix	$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\mathbf{u} = \mathbf{u}^T\mathbf{v}^T$	$\mathbf{u} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$ $\mathbf{v} = [5 \quad 4 \quad 0]$	$\mathbf{v}\mathbf{u} = [5 \quad 4 \quad 0]\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$ $\mathbf{u}^T\mathbf{v}^T = [1 \quad -3 \quad 5]\begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$
u a row matrix and **v** a row matrix	$\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\mathbf{v}^T = \mathbf{v}\mathbf{u}^T$	$\mathbf{u} = [1 \quad -3 \quad 5]$ $\mathbf{v} = [5 \quad 4 \quad 0]$	$\mathbf{u}\mathbf{v}^T = [1 \quad -3 \quad 5]\begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7$ $\mathbf{v}\mathbf{u}^T = [5 \quad 4 \quad 0]\begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$

and therefore for a n × n matrix A

$Au\,.\,v = v^TAu = u \cdot A^Tv$

$u\,.\,Av = v^TA^Tu = A^T u\,.\,v$

EXAMPLE: Verifying that $Au \cdot v = u \cdot A^Tv$



Suppose that

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$$

Then

$$A\mathbf{u} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix}$$

$$A^T\mathbf{v} = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 4 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ -1 \end{bmatrix}$$

from which we obtain

$$A\mathbf{u} \cdot \mathbf{v} = 7(-2) + 10(0) + 5(5) = 11$$
$$\mathbf{u} \cdot A^T\mathbf{v} = (-1)(-7) + 2(4) + 4(-1) = 11$$

Example: $Let \quad u = (-3, 2, 1, 0), \quad v = (4, 7, -3, 2), \quad w = (5, -2, 8, 1)$.

Find:

a) $v - w$= (-1, 9, -11, 1)

b) 2u+7v= (-6, 4, 2, 0) + (28, 49, -21, 14) = (22, 53, -19, 14)

c) –u + (v – 4w) = (3, -2, -1, 0) + (-16, 15, -35, -2) = (-13, 13, -36, -2)

d) $\|u\| = \sqrt{9+4+1+0} = \sqrt{14}$

e) $\|u - v\| = \sqrt{49+25+16+4} = \sqrt{94}$

f) $\|u\| + \|v\| = \sqrt{14} + \sqrt{16+49+9+4} = \sqrt{14} + \sqrt{78}$

g) $\dfrac{w}{\|w\|} = \dfrac{(5,-2,8,1)}{\sqrt{25+4+64+1}} = (\dfrac{5}{\sqrt{94}}, \dfrac{-2}{\sqrt{94}}, \dfrac{8}{\sqrt{94}}, \dfrac{1}{\sqrt{94}})$

h) $\left\| \dfrac{w}{\|w\|} \right\| = \sqrt{\dfrac{25}{94} + \dfrac{4}{94} + \dfrac{64}{94} + \dfrac{1}{94}} = \sqrt{\dfrac{94}{94}} = 1$

Example: **Find** all values of k for which the following vectors are orthogonal

u=(k, k, 1), v = (k, 5, 6).

Solution: Since u and v are orthogonal then u . v = 0

k^2+5k+6 = 0

(k+3) (k+2) = 0

k= -3, k=-2

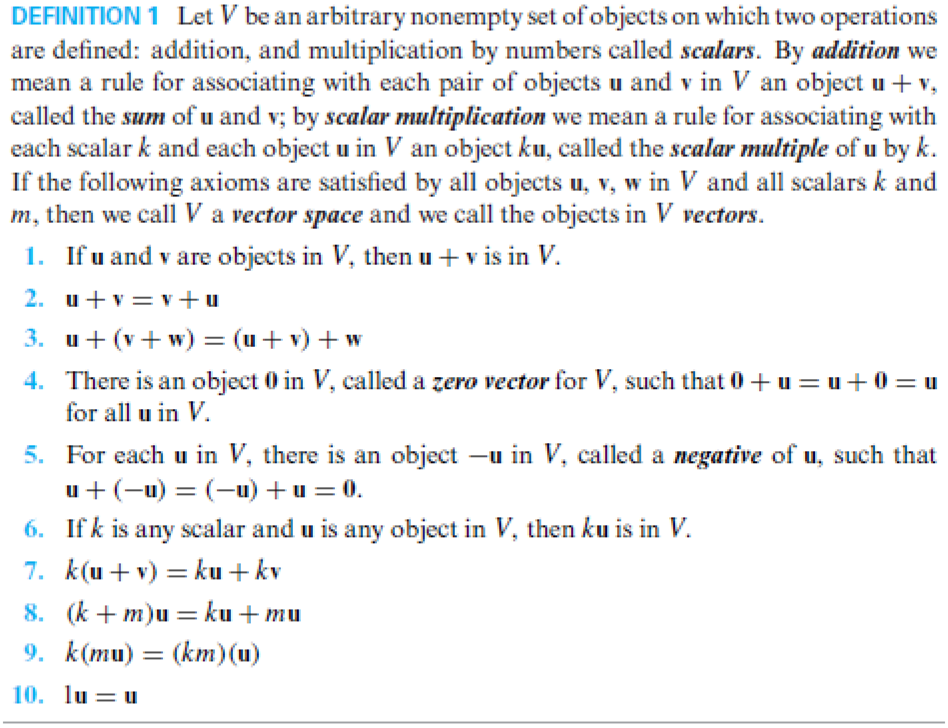
Example: Find u . v where $\|u+v\|=1, \quad \|u-v\|=5$

Solution: $u \,.\, v=\frac{1}{4}\|u+v\|^{2}-\frac{1}{4}\|u-v\|^{2}$

$$u \,.\, v=\frac{1}{4}(1)-\frac{1}{4}(25)=\frac{-24}{4}=-6$$

General Vector Spaces

4.1 Real Vector Spaces



DEFINITION 1 Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by numbers called ***scalars***. By ***addition*** we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u}+\mathbf{v}$, called the ***sum*** of $\mathbf{u}$ and $\mathbf{v}$; by ***scalar multiplication*** we mean a rule for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the ***scalar multiple*** of $\mathbf{u}$ by k. If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and all scalars k and m, then we call V a ***vector space*** and we call the objects in V ***vectors***.

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
4. There is an object $\mathbf{0}$ in V, called a ***zero vector*** for V, such that $\mathbf{0}+\mathbf{u}=\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}$ in V.
5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a ***negative*** of $\mathbf{u}$, such that $\mathbf{u}+(-\mathbf{u})=(-\mathbf{u})+\mathbf{u}=\mathbf{0}$.
6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.
7. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
8. $(k+m)\mathbf{u}=k\mathbf{u}+m\mathbf{u}$
9. $k(m\mathbf{u})=(km)(\mathbf{u})$
10. $1\mathbf{u}=\mathbf{u}$

To Show That a Set with Two Operations Is a Vector Space

Step 1. Identify the set *V* of objects that will become vectors.

Step 2. Identify the addition and scalar multiplication operations on V.

Step 3. Verify Axioms 1 and 6; that is, adding two vectors in V produces a vector in V, and multiplying a vector in V by a scalar also produces a vector in V.

Axiom 1 is called ***closure under addition***, and Axiom 6 is called ***closure under scalar multiplication***.

Step 4. Confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold.

EXAMPLE: The Zero Vector Space

Let V consist of a single object, which we denote by $\mathbf{0}$, and define

$\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $k\,\mathbf{0} = \mathbf{0}$

for all scalars k. It is easy to check that all the vector space axioms are satisfied. We call this the ***zero vector space***.

EXAMPLE: R^n Is a Vector Space

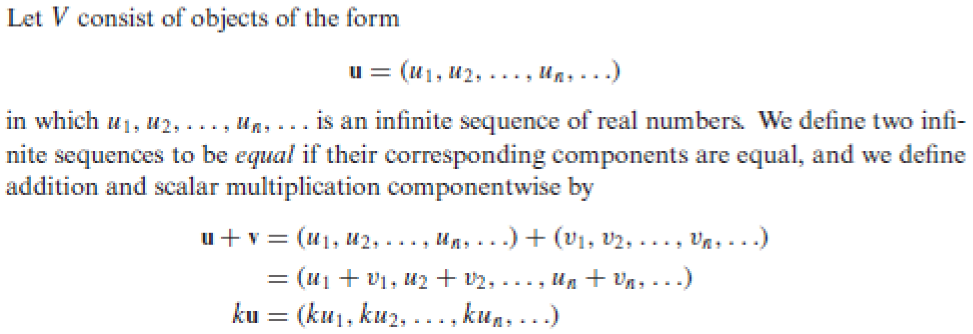
Let $V = R^n$, and define the vector space operations on V to be the usual operations of addition and scalar multiplication of n-tuples; that is,

$\mathbf{u} + \mathbf{v} = (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$

$k\mathbf{u} = (ku_1, ku_2, \ldots, ku_n)$

The set $V = R^n$ is closed under addition and scalar multiplication because the foregoing operations produce n-tuples as their end result, and these operations satisfy Axioms 2,3, 4, 5, 7, 8, 9, and 10 by virtue of Theorem.

EXAMPLE: The Vector Space of Infinite Sequences of Real Numbers



Let V consist of objects of the form

$$\mathbf{u} = (u_1, u_2, \ldots, u_n, \ldots)$$

in which $u_1, u_2, \ldots, u_n, \ldots$ is an infinite sequence of real numbers. We define two infinite sequences to be *equal* if their corresponding components are equal, and we define addition and scalar multiplication componentwise by

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (u_1, u_2, \ldots, u_n, \ldots) + (v_1, v_2, \ldots, v_n, \ldots) \\ &= (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n, \ldots) \\ k\mathbf{u} &= (ku_1, ku_2, \ldots, ku_n, \ldots) \end{aligned}$$

In the exercises we ask you to confirm that V with these operations is a vector space. We will denote this vector space by the symbol R^∞.

EXAMPLE: The Vector Space of 2 × 2 Matrices

Let V be the set of 2×2 matrices with real entries, and take the vector space operations on V to be the usual operations of matrix addition and scalar multiplication; that is,

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}=\begin{bmatrix}u_{11}+v_{11} & u_{12}+v_{12}\\ u_{21}+v_{21} & u_{22}+v_{22}\end{bmatrix} \tag{1}$$

$$k\mathbf{u}=k\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}=\begin{bmatrix}ku_{11} & ku_{12}\\ ku_{21} & ku_{22}\end{bmatrix}$$

The set V is closed under addition and scalar multiplication because the foregoing operations produce 2×2 matrices as the end result. Thus, it remains to confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold. Some of these are standard properties of matrix operations. For example, Axiom 2 follows from Theorem 1.4.1(a) since

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}=\begin{bmatrix}v_{11} & v_{12}\\ v_{21} & v_{22}\end{bmatrix}+\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}=\mathbf{v}+\mathbf{u}$$

Similarly, Axioms 3, 7, 8, and 9 follow from parts (b), (h), (j), and (e), respectively, of that theorem (verify). This leaves Axioms 4, 5, and 10 that remain to be verified.

To confirm that Axiom 4 is satisfied, we must find a 2×2 matrix $\mathbf{0}$ in V for which $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}$ for all 2×2 matrices in V. We can do this by taking

$$\mathbf{0}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

With this definition,

$$\mathbf{0}+\mathbf{u}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}=\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}=\mathbf{u}$$

and similarly $\mathbf{u}+\mathbf{0}=\mathbf{u}$. To verify that Axiom 5 holds we must show that each object $\mathbf{u}$ in V has a negative $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$ and $(-\mathbf{u})+\mathbf{u}=\mathbf{0}$. This can be done by defining the negative of $\mathbf{u}$ to be

$$-\mathbf{u}=\begin{bmatrix}-u_{11} & -u_{12}\\ -u_{21} & -u_{22}\end{bmatrix}$$

With this definition,

$$\mathbf{u}+(-\mathbf{u})=\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}+\begin{bmatrix}-u_{11} & -u_{12}\\ -u_{21} & -u_{22}\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\mathbf{0}$$

and similarly $(-\mathbf{u})+\mathbf{u}=\mathbf{0}$. Finally, Axiom 10 holds because

$$1\mathbf{u}=1\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}=\begin{bmatrix}u_{11} & u_{12}\\ u_{21} & u_{22}\end{bmatrix}=\mathbf{u}$$

EXAMPLE: let V= $\{(x,y,z)\in R^3$: ax+by+cz= 0, a,b,c$\in$R$\}$ is a vector space

Let u,v $\in$ V

u=(x_1, y_1, z_1) s.t $ax_1+by_1+cz_1=0$

v=(x_2, y_2, z_2) s.t $ax_2+by_2+cz_2=0$

u+v= $(x_1+x_2, y_1+y_2, z_1+z_2)$

$a(x_1+x_2)+b(y_1+y_2)+c(z_1+z_2) \overset{?}{=} 0$

$a(x_1+x_2)+b(y_1+y_2)+c(z_1+z_2) = ax_1+ ax_2 +by_1+by_2+cz_1+cz_2$

= ax1+by1+cz1+ ax2+by2+cz2 = 0+0

Then $u+v \in V$

$ku = (kx_1, k y_1, k z_1)$

$k\, ax_1 + kby_1 + kcz_1 \overset{?}{=} 0$

$k\, ax_1 + kby_1 + kcz_1 = k(\ ax_1 + by_1 + cz_1) = k \,.\, 0 = 0$

Then $ku \in V$

EXAMPLE: let $V=\{(x,y) \in R^2 : x \geq o\}$ is not a vector space

$u=(x_1,y_1)$, $x_1 \geq o$

$v=(x_2,y_2)$, $x_2 \geq o$

$u+v= (x_1+x_2 , y_1+y_2)$, $x_1+x_2 \geq o$

$k.u=(kx_1,ky_1)$, if $k=-10$ then $kx_1 \leq 0$

4.2 Subspaces

THEOREM *If W is a set of one or more vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.*
(*a*) *If* **u** *and* **v** *are vectors in W, then* **u** + **v** *is in W.*
(*b*) *If k is a scalar and* **u** *is a vector in W, then k***u** *is in W.*

EXAMPLE: The Zero Subspace

If *V* is any vector space, and if *W* = {**0**} is the subset of *V* that consists of the zero vector only, then *W* is closed under addition and scalar multiplication since

0 + **0** = **0** and *k***0** = **0**

for any scalar *k*. We call *W* the ***zero subspace*** of *V*.

EXAMPLE: A Subset of R^2 That Is Not a Subspace

Let *W* be the set of all points *(x, y)* in R^2 for which $x \geq 0$ and $y \geq 0$. This set is not a subspace of R^2 because it is not closed under scalar multiplication. For example, **v** = *(*1*,* 1*)* is a vector in *W*, but *(*−1*)***v** = *(*−1*,*−1*)* is not.

EXAMPLE: Subspaces of M_{nn}

We know from Theorem that the sum of two symmetric $n \times n$ matrices is symmetric and that a scalar multiple of a symmetric $n \times n$ matrix is symmetric. Thus, the set of symmetric $n \times n$ matrices is closed under addition and scalar multiplication and hence is a subspace of M_{nn}. Similarly, the sets of upper triangular matrices, lower triangular matrices, and diagonal matrices are subspaces of M_{nn}.

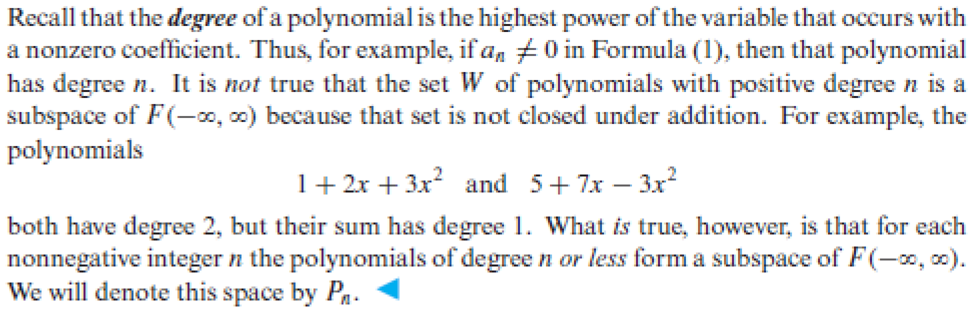
EXAMPLE: A Subset of M_{nn} That Is Not a Subspace

The set W of invertible $n \times n$ matrices is not a subspace of M_{nn}, failing on two counts—it is not closed under addition and not closed under scalar multiplication. We will illustrate this with an example in M_{22} that you can readily adapt to M_{nn}. Consider the matrices

$$U = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \text{ and } V = \begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix} \quad U + V = \begin{bmatrix} 0 & 4 \\ 0 & 10 \end{bmatrix}$$

The matrix $0U$ is the 2×2 zero matrix and hence is not invertible, and the matrix $U + V$ has a column of zeros so it also is not invertible.

EXAMPLE: The Subspace of Polynomials of Degree $\leq n$



Recall that the ***degree*** of a polynomial is the highest power of the variable that occurs with a nonzero coefficient. Thus, for example, if $a_n \neq 0$ in Formula (1), then that polynomial has degree n. It is *not* true that the set W of polynomials with positive degree n is a subspace of $F(-\infty, \infty)$ because that set is not closed under addition. For example, the polynomials

$$1 + 2x + 3x^2 \quad \text{and} \quad 5 + 7x - 3x^2$$

both have degree 2, but their sum has degree 1. What *is* true, however, is that for each nonnegative integer n the polynomials of degree n *or less* form a subspace of $F(-\infty, \infty)$. We will denote this space by P_n. ◀

DEFINITION: If $\mathbf{w}$ is a vector in a vector space V, then $\mathbf{w}$ is said to be a ***linear combination*** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in V if $\mathbf{w}$ can be expressed in the form

$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$

where $k_1, k_2, \ldots, k_r$ are scalars. These scalars are called the ***coefficients*** of the linear combination.

THEOREM: If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then:

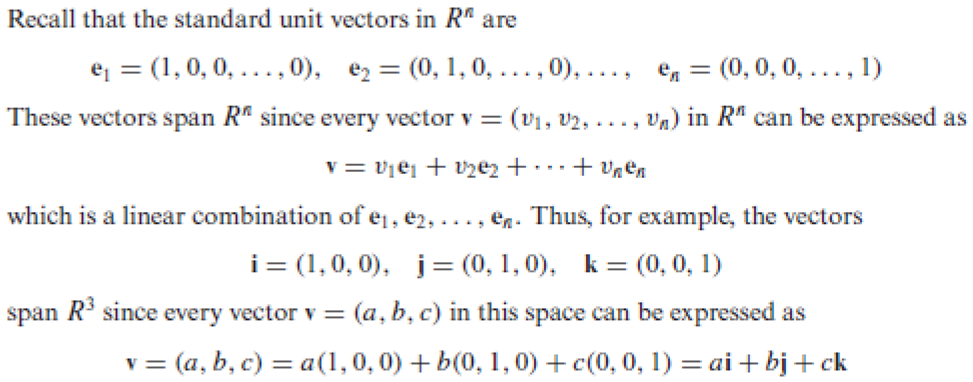
(a) The set W of all possible linear combinations of the vectors in S is a subspace of V.

(b) The set W in part (a) is the “smallest” subspace of V that contains all of the vectors in S in the sense that any other subspace that contains those vectors contains W.

DEFINITION: If S = {$\mathbf{w}_1$, $\mathbf{w}_2$, . . . , $\mathbf{w}_r$ } is a nonempty set of vectors in a vector space V , then the subspace W of V that consists of all possible linear combinations of the vectors in S is called the subspace of V **generated** by S, and we say that the vectors $\mathbf{w}_1$, $\mathbf{w}_2$, . . . , $\mathbf{w}_r$ **span** W. We denote this subspace as

W = span{$\mathbf{w}_1$, $\mathbf{w}_2$, . . . , $\mathbf{w}_r$ } or W = span(S)

EXAMPLE: The Standard Unit Vectors Span *R^n*



Recall that the standard unit vectors in R^n are

$$\mathbf{e}_1 = (1, 0, 0, \dots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \quad \mathbf{e}_n = (0, 0, 0, \dots, 1)$$

These vectors span R^n since every vector $\mathbf{v} = (v_1, v_2, \dots, v_n)$ in R^n can be expressed as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n$$

which is a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$. Thus, for example, the vectors

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1)$$

span R^3 since every vector $\mathbf{v} = (a, b, c)$ in this space can be expressed as

$$\mathbf{v} = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

R^n = span {e_1, e_2, ……., e_n }

R^3= span{ e_1, e_2,e_3}

u=(2, 3, -2)

u=$2e_1$+$3e_2$+-$2e_3$

=(2,0,0)+(0,3,0)+(0,0,-2)=(2,3,-2)

V=(5,0,0)= $5e_1$

EXAMPLE: A Spanning Set for P_n

The polynomials $1, x, x^2, \ldots, x^n$ span the vector space P_n defined in Example

each polynomial **p** in P_n can be written as

$$\mathbf{p} = a_0 + a_1x + \cdots + a_nx^n$$

which is a linear combination of $1, x, x^2, \ldots, x^n$. We can denote this by writing

$$P_n = \operatorname{span}\{1, x, x^2, \ldots, x^n\} \blacktriangleleft$$

$1+x^{10}=1+0x+0x^2+0+0+\ldots.+1x^{10}$

P_3= span{ 1, x, x^2,x^3}

EXAMPLE: Linear Combinations

Consider the vectors $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (6, 4, 2)$ in R^3. Show that $\mathbf{w} = (9, 2, 7)$ is a linear combination of **u** and **v** and that $\mathbf{w}' = (4, -1, 8)$ is *not* a linear combination of **u** and **v**.

Solution In order for **w** to be a linear combination of **u** and **v**, there must be scalars k_1 and k_2 such that $\mathbf{w} = k_1\mathbf{u} + k_2\mathbf{v}$; that is,

$$(9, 2, 7) = k_1(1, 2, -1) + k_2(6, 4, 2) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

Equating corresponding components gives

$$\begin{aligned} k_1 + 6k_2 &= 9 \\ 2k_1 + 4k_2 &= 2 \\ -k_1 + 2k_2 &= 7 \end{aligned}$$

Solving this system using Gaussian elimination yields $k_1 = -3$, $k_2 = 2$, so

$$\mathbf{w} = -3\mathbf{u} + 2\mathbf{v}$$

Similarly, for $\mathbf{w}'$ to be a linear combination of **u** and **v**, there must be scalars k_1 and k_2 such that $\mathbf{w}' = k_1\mathbf{u} + k_2\mathbf{v}$; that is,

$$(4, -1, 8) = k_1(1, 2, -1) + k_2(6, 4, 2) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

Equating corresponding components gives

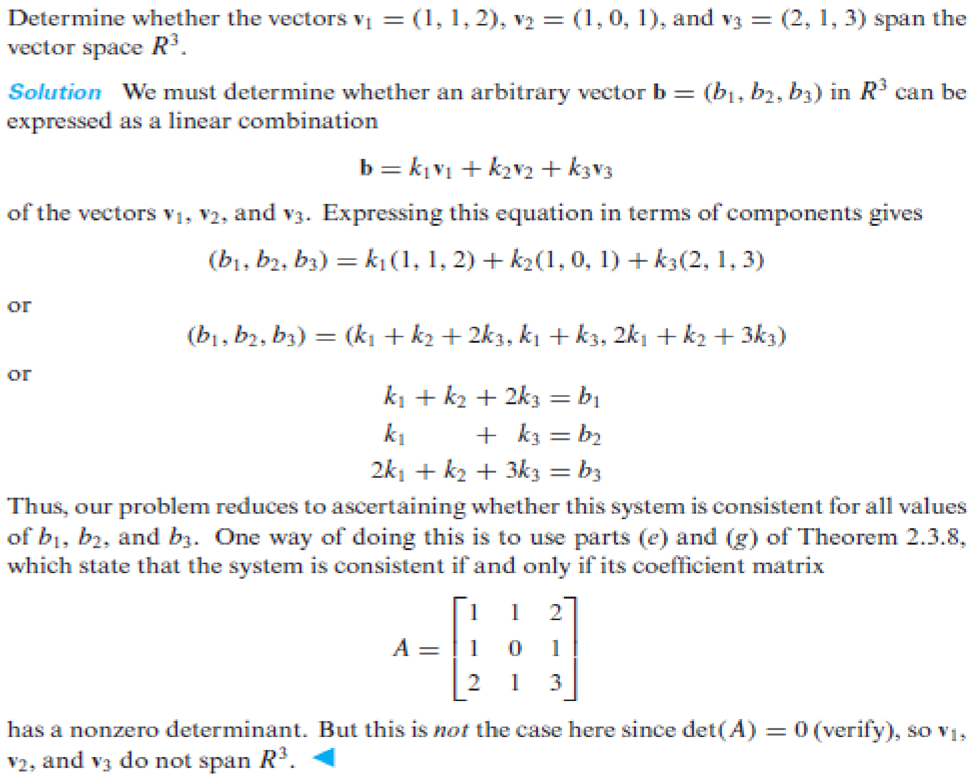
$$\begin{aligned} k_1 + 6k_2 &= 4 \\ 2k_1 + 4k_2 &= -1 \\ -k_1 + 2k_2 &= 8 \end{aligned}$$

This system of equations is inconsistent (verify), so no such scalars k_1 and k_2 exist. Consequently, $\mathbf{w}'$ is not a linear combination of **u** and **v**.

$8k_2=12$

$8k_2=15$

EXAMPLE: Testing for Spanning



Determine whether the vectors $\mathbf{v}_1 = (1, 1, 2)$, $\mathbf{v}_2 = (1, 0, 1)$, and $\mathbf{v}_3 = (2, 1, 3)$ span the vector space R^3.

Solution We must determine whether an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ in R^3 can be expressed as a linear combination

$$\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$$

of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Expressing this equation in terms of components gives

$$(b_1, b_2, b_3) = k_1(1, 1, 2) + k_2(1, 0, 1) + k_3(2, 1, 3)$$

or

$$(b_1, b_2, b_3) = (k_1 + k_2 + 2k_3,\ k_1 + k_3,\ 2k_1 + k_2 + 3k_3)$$

or

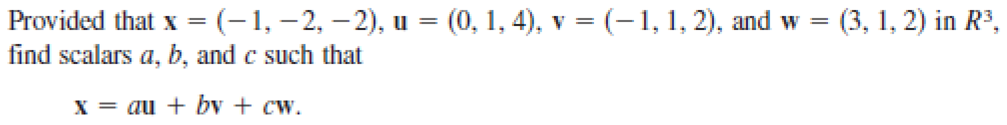
$$\begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 \qquad + \ k_3 &= b_2 \\ 2k_1 + k_2 + 3k_3 &= b_3 \end{aligned}$$

Thus, our problem reduces to ascertaining whether this system is consistent for all values of b_1, b_2, and b_3. One way of doing this is to use parts (*e*) and (*g*) of Theorem 2.3.8, which state that the system is consistent if and only if its coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

has a nonzero determinant. But this is *not* the case here since $\det(A) = 0$ (verify), so $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ do not span R^3. ◀

EXAMPLE:

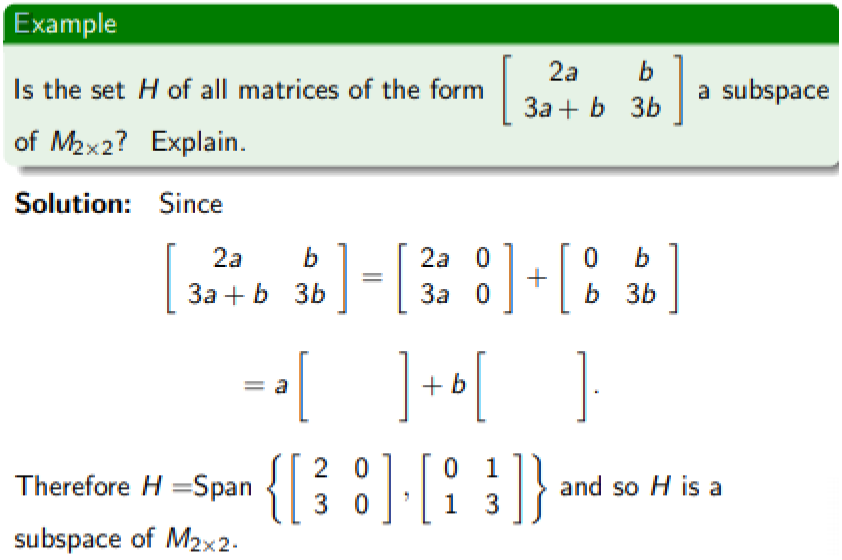


Provided that $\mathbf{x} = (-1, -2, -2)$, $\mathbf{u} = (0, 1, 4)$, $\mathbf{v} = (-1, 1, 2)$, and $\mathbf{w} = (3, 1, 2)$ in R^3, find scalars a, b, and c such that

$$\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}.$$

$(-1, -2, -2)= a(0, 1, 4)+b(-1, 1, 2)+c(3, 1, 2)$

$-b+3c= -1$,$a+b+c= -2$,$4a+2b+2c= -2$



Example

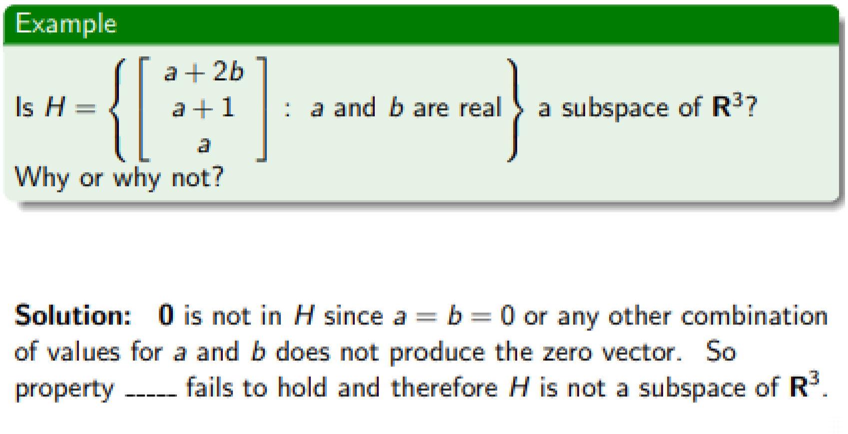
Is the set H of all matrices of the form $\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix}$ a subspace of $M_{2\times 2}$? Explain.

Solution: Since

$$\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 3a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 3b \end{bmatrix}$$

$$= a\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} + b\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}.$$

Therefore H =Span $\left\{ \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \right\}$ and so H is a subspace of $M_{2\times 2}$.



Example

Is $H = \left\{ \begin{bmatrix} a+2b \\ a+1 \\ a \end{bmatrix} : a \text{ and } b \text{ are real} \right\}$ a subspace of $\mathbf{R}^3$?

Why or why not?

Solution: $\mathbf{0}$ is not in H since $a = b = 0$ or any other combination of values for a and b does not produce the zero vector. So property _____ fails to hold and therefore H is not a subspace of $\mathbf{R}^3$.

Exercise: In each part, determine whether the vectors span R^3.

(a) $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 1, 0)$, $\mathbf{v}_3 = (1,0, 0)$

(b) $\mathbf{v}_1 = (2,-1, 3)$, $\mathbf{v}_2 = (4, 1, 2)$, $\mathbf{v}_3 = (8,-1, 8)$

Solution:

(a) Is R^3= span { v_1, v_2, v_3}

Any vector in R^3 has the form

(a, b, c)= $k_1v_1 + k_2 v_2 + k_3 v_3$

= $k_1(1, 1, 1) + k_2 (1, 1, 0) + k_3 (0,0, 0)$

= $(k_1+ k_2+ k_3 , k_1+ k_2 , k_1)$

a= $k_1+ k_2+ k_3$

b= $k_1+ k_2$

c= k_1

then k_1= c , k_2 = b-c , k_3 = a-b

THEOREM: *The solution set of a homogeneous linear system A***x** = **0** *of m equations in n unknowns is a subspace of* R^n .

EXAMPLE: Solution Spaces of Homogeneous Systems

In each part, solve the system by any method and then give a geometric description of the solution set.

(a) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution

(a) The solutions are

$$x = 2s - 3t, \quad y = s, \quad z = t$$

from which it follows that

$$x = 2y - 3z \quad \text{or} \quad x - 2y + 3z = 0$$

This is the equation of a plane through the origin that has $\mathbf{n} = (1, -2, 3)$ as a normal.

(b) The solutions are

$$x = -5t, \quad y = -t, \quad z = t$$

which are parametric equations for the line through the origin that is parallel to the vector $\mathbf{v} = (-5, -1, 1)$.

(c) The only solution is $x = 0$, $y = 0$, $z = 0$, so the solution space consists of the single point $\{\mathbf{0}\}$.

(d) This linear system is satisfied by all real values of x, y, and z, so the solution space is all of R^3. ◀

4.3 Linear Independence

DEFINITION 1 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a set of two or more vectors in a vector space V, then S is said to be a ***linearly independent set*** if no vector in S can be expressed as a linear combination of the others. A set that is not linearly independent is said to be ***linearly dependent***.

THEOREM 4.3.1 *A nonempty set* $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ *in a vector space* V *is linearly independent if and only if the only coefficients satisfying the vector equation*

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

are $k_1 = 0, k_2 = 0, \ldots, k_r = 0$.

EXAMPLE: Linear Independence in R^3

Determine whether the vectors

$\mathbf{v}_1 = (1,-2, 3), \mathbf{v}_2 = (5, 6,-1), \mathbf{v}_3 = (3, 2, 1)$

are linearly independent or linearly dependent in R^3.

Solution The linear independence or dependence of these vectors is determined by whether the vector equation

$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$

can be satisfied with coefficients that are not all zero.

$k_1(1,-2, 3) + k_2(5, 6,-1) + k_3(3, 2, 1) = (0, 0, 0)$

Equating corresponding components on the two sides yields the homogeneous linear System

$$k_1 + 5k_2 + 3k_3 = 0$$

$$-2k_1 + 6k_2 + 2k_3 = 0$$

$$3k_1 - k_2 + k_3 = 0$$

Thus, our problem reduces to determining whether this system has nontrivial solutions. There are various ways to do this; one possibility is to simply solve the system, which yields

$k_1 = -\frac{1}{2}t,\ k_2 = -\frac{1}{2}t,\ k_3 = t$

This shows that the system has nontrivial solutions and hence that the vectors are linearly dependent. A second method for establishing the linear dependence is to take advantage of the fact that the coefficient matrix

$$A = \begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix} \ \det(A) = 0$$

from which it follows that has nontrivial solutions.

EXAMPLE: Linear Independence in R^4

Determine whether the vectors

$$\mathbf{v}_1 = (1, 2, 2, -1), \quad \mathbf{v}_2 = (4, 9, 9, -4), \quad \mathbf{v}_3 = (5, 8, 9, -5)$$

in R^4 are linearly dependent or linearly independent.

Solution The linear independence or linear dependence of these vectors is determined by whether there exist nontrivial solutions of the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$$

or, equivalently, of

$$k_1(1, 2, 2, -1) + k_2(4, 9, 9, -4) + k_3(5, 8, 9, -5) = (0, 0, 0, 0)$$

Equating corresponding components on the two sides yields the homogeneous linear system

$$\begin{aligned} k_1 + 4k_2 + 5k_3 &= 0 \\ 2k_1 + 9k_2 + 8k_3 &= 0 \\ 2k_1 + 9k_2 + 9k_3 &= 0 \\ -k_1 - 4k_2 - 5k_3 &= 0 \end{aligned}$$

We leave it for you to show that this system has only the trivial solution

$$k_1 = 0, \quad k_2 = 0, \quad k_3 = 0$$

from which you can conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent.

EXAMPLE: Linear Independence of Polynomials

Determine whether the polynomials

$$\mathbf{p}_1 = 1 - x, \quad \mathbf{p}_2 = 5 + 3x - 2x^2, \quad \mathbf{p}_3 = 1 + 3x - x^2$$

are linearly dependent or linearly independent in P_2.

Solution The linear independence or dependence of these vectors is determined by whether the vector equation

$$k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 = \mathbf{0}$$

$$k_1(1 - x) + k_2(5 + 3x - 2x^2) + k_3(1 + 3x - x^2) = 0$$

$$(k_1 + 5k_2 + k_3) + (-k_1 + 3k_2 + 3k_3)x + (-2k_2 - k_3)x^2 = 0$$

Since this equation must be satisfied by all x in $(-\infty, \infty)$, each coefficient must be zero (as explained in the previous example). Thus, the linear dependence or independence of the given polynomials hinges on whether the following linear system has a nontrivial solution:

$$\begin{aligned} k_1 + 5k_2 + k_3 &= 0 \\ -k_1 + 3k_2 + 3k_3 &= 0 \\ - 2k_2 - k_3 &= 0 \end{aligned}$$

We leave it for you to show that this linear system has nontrivial solutions either by solving it directly or by showing that the coefficient matrix has determinant zero. Thus, the set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly dependent. ◀

THEOREM: Let S = {$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$} be a set of vectors in R^n. Ifr > n, then S is linearly dependent

DEFINITION 2 If $\mathbf{f}_1 = f_1(x), \mathbf{f}_2 = f_2(x), \ldots, \mathbf{f}_n = f_n(x)$ are functions that are $n-1$ times differentiable on the interval $(-\infty, \infty)$, then the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the ***Wronskian*** of $f_1, f_2, \ldots, f_n$.

THEOREM: If the functions $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$ have n−1 continuous derivatives on the interval (−∞, ∞), and if the Wronskian of these functions is not identically zero on (−∞, ∞), then these functions form a linearly independent set of vectors in $C^{(n-1)}$ (−∞, ∞).

EXAMPLE: Linear Independence Using the Wronskian

Use the Wronskian to show that $\mathbf{f}_1 = x$ and $\mathbf{f}_2 = \sin x$ are linearly independent vectors in $C^{(n-1)}$ (−∞, ∞).

Solution The Wronskian is

$$W(x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x \cos x - \sin x$$

This function is not identically zero on the interval $(-\infty, \infty)$ since, for example,

$$W\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Thus, the functions are linearly independent.

EXAMPLE: Linear Independence Using the Wronskian

Use the Wronskian to show that $\mathbf{f}_1 = 1$, $\mathbf{f}_2 = e^x$, and $\mathbf{f}_3 = e^{2x}$ are linearly independent vectors in $C^\infty(-\infty, \infty)$.

Solution The Wronskian is

$$W(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} = 2e^{3x}$$

This function is obviously not identically zero on $(-\infty, \infty)$, so $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$ form a linearly independent set. ◄

4.4 Basis and dimension

Basis for a Vector Space

DEFINITION: If S = {$\mathbf{v}_1$, $\mathbf{v}_2$, . . . , vn } is a set of vectors in a finite-dimensional vector space V , then S is called a **basis** for V if:

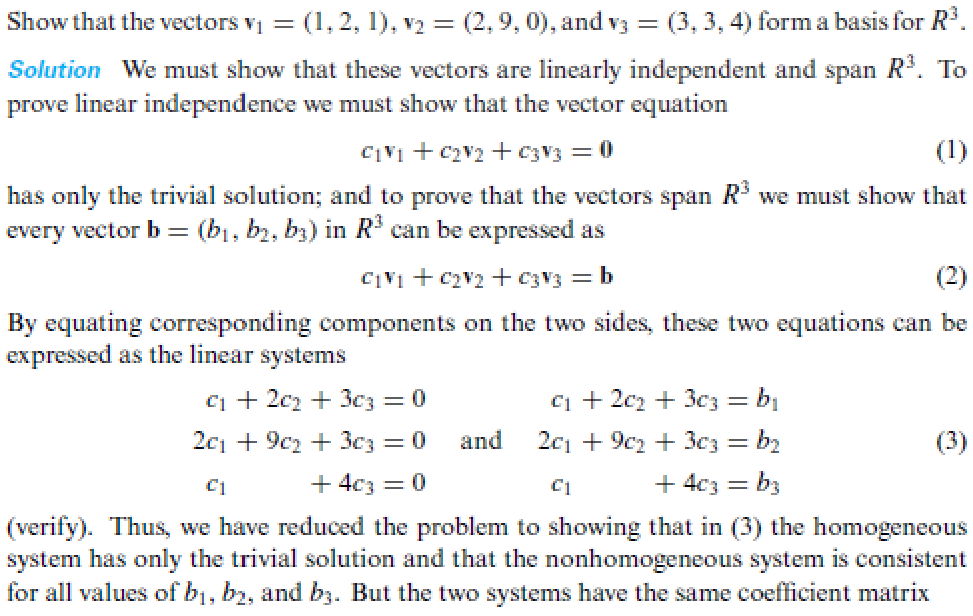
(a) S spans V.

(b) S is linearly independent.

S = {1, x, x^2, . . . , x^n} is a basis for the vector space P_n.

e_1 = (1, 0, 0, . . . , 0), e_2 = (0, 1, 0, . . . , 0), . . . , e_n = (0, 0, 0, . . . , 1) The Standard Basis for R^n.

EXAMPLE:



Show that the vectors $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 9, 0)$, and $\mathbf{v}_3 = (3, 3, 4)$ form a basis for R^3.

Solution We must show that these vectors are linearly independent and span R^3. To prove linear independence we must show that the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \tag{1}$$

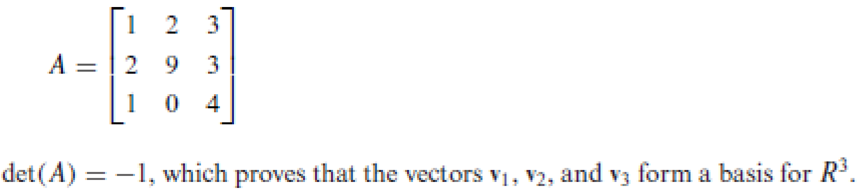
has only the trivial solution; and to prove that the vectors span R^3 we must show that every vector $\mathbf{b} = (b_1, b_2, b_3)$ in R^3 can be expressed as

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b} \tag{2}$$

By equating corresponding components on the two sides, these two equations can be expressed as the linear systems

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ 2c_1 + 9c_2 + 3c_3 &= 0 \\ c_1 \qquad\quad + 4c_3 &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} c_1 + 2c_2 + 3c_3 &= b_1 \\ 2c_1 + 9c_2 + 3c_3 &= b_2 \\ c_1 \qquad\quad + 4c_3 &= b_3 \end{aligned} \tag{3}$$

(verify). Thus, we have reduced the problem to showing that in (3) the homogeneous system has only the trivial solution and that the nonhomogeneous system is consistent for all values of b_1, b_2, and b_3. But the two systems have the same coefficient matrix



$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$$

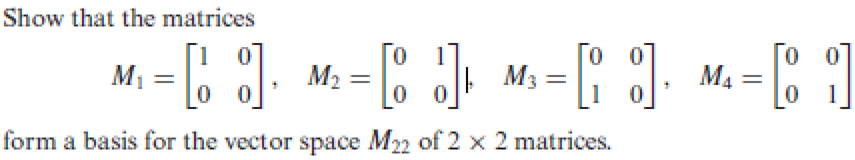
$\det(A) = -1$, which proves that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for R^3.

S={v_1,v_2,v_3}

$3v_1+4v_2-v_3$ = (3,6,3)+(8,36,0)-(3,3,4) =(8, 39, -1)

v=(8, 39, -1) (v)s=(3, 4, -1)

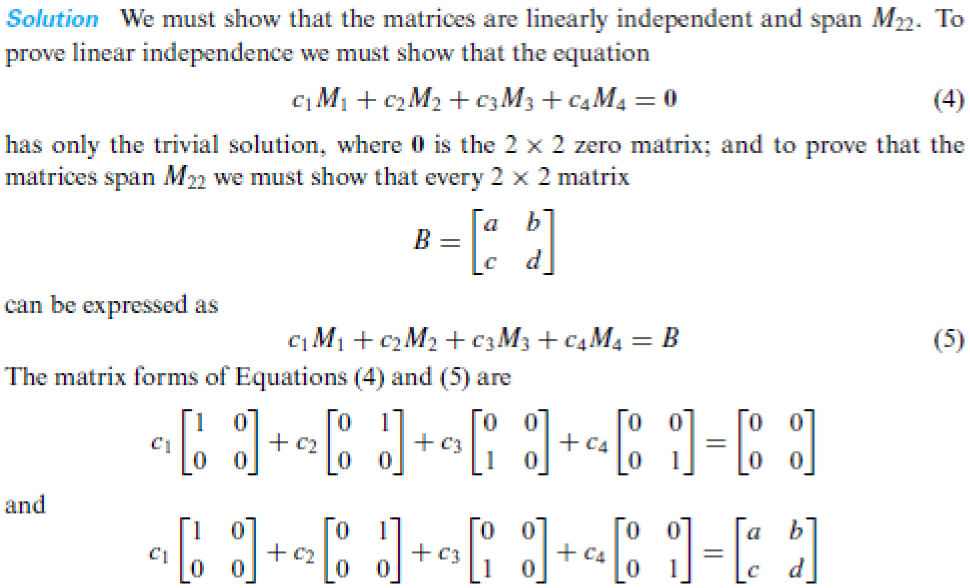
EXAMPLE:



Show that the matrices

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the vector space M_{22} of 2 × 2 matrices.



Solution We must show that the matrices are linearly independent and span M_{22}. To prove linear independence we must show that the equation

$$c_1M_1 + c_2M_2 + c_3M_3 + c_4M_4 = \mathbf{0} \tag{4}$$

has only the trivial solution, where $\mathbf{0}$ is the 2 × 2 zero matrix; and to prove that the matrices span M_{22} we must show that every 2 × 2 matrix

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

can be expressed as

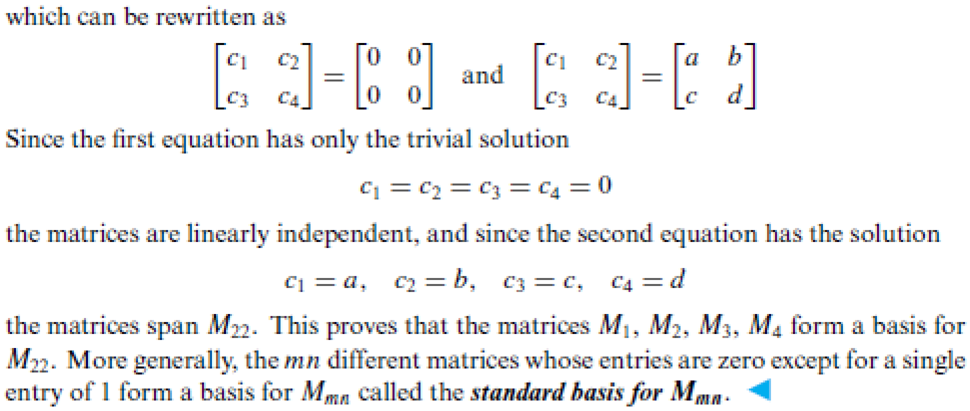
$$c_1M_1 + c_2M_2 + c_3M_3 + c_4M_4 = B \tag{5}$$

The matrix forms of Equations (4) and (5) are

$$c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$



which can be rewritten as

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Since the first equation has only the trivial solution

$$c_1 = c_2 = c_3 = c_4 = 0$$

the matrices are linearly independent, and since the second equation has the solution

$$c_1 = a, \quad c_2 = b, \quad c_3 = c, \quad c_4 = d$$

the matrices span M_{22}. This proves that the matrices M_1, M_2, M_3, M_4 form a basis for M_{22}. More generally, the mn different matrices whose entries are zero except for a single entry of 1 form a basis for M_{mn} called the ***standard basis for*** $\boldsymbol{M_{mn}}$. ◄

DEFINITION: If S = {v_1, v_2, . . . , v_n} is a basis for a vector space V, and

$$v = c_1v_1 + c_2v_2 + \cdot\cdot\cdot + c_nv_n$$

is the expression for a vector v in terms of the basis S, then the scalars c_1, c_2, . . . , c_n are called the coordinates of v relative to the basis S. The vector (c_1, c_2, . . . , c_n) in R^n constructed from these coordinates is called the coordinate vector of v relative to S; it is denoted by $(v)_S$ = (c_1, c_2, . . . , c_n)

EXAMPLE:

(a) The vectors $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (2, 9, 0)$, $\mathbf{v}_3 = (3, 3, 4)$ form a basis for R^3. Find the coordinate vector of $\mathbf{v} = (5,-1, 9)$ relative to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(b) Find the vector $\mathbf{v}$ in R^3 whose coordinate vector relative to S is $(\mathbf{v})_S = (-1, 3, 2)$.

Solution (a) To find $(\mathbf{v})_S$ we must first express $\mathbf{v}$ as a linear combination of the vectors in S; that is, we must find values of c_1, c_2, and c_3 such that

$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

or, in terms of components,

$(5,-1, 9) = c_1(1, 2, 1) + c_2(2, 9, 0) + c_3(3, 3, 4)$

Equating corresponding components gives

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 5 \\ 2c_1 + 9c_2 + 3c_3 &= -1 \\ c_1 + 4c_3 &= 9 \end{aligned}$$

Solving this system we obtain $c_1 = 1$, $c_2 = -1$, $c_3 = 2$. Therefore, $(\mathbf{v})_S = (1,-1, 2)$

Solution (b) Using the definition of $(\mathbf{v})_S$, we obtain

$\mathbf{v} = (-1)\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3$

$= (-1)(1, 2, 1) + 3(2, 9, 0) + 2(3, 3, 4) = (11, 31, 7)$

DEFINITION: The **dimension** of a finite-dimensional vector space V is denoted by dim (V) and is defined to be the number of vectors in a basis for V. In addition, the zero vector space is defined to have dimension zero.

EXAMPLE: Dimensions of Some Familiar Vector Spaces

$\dim(R^n) = n$ **[The standard basis has n vectors.]**

$\dim(P_n) = n + 1$ **[The standard basis has n + 1 vectors.]**

$\dim(M_{mn}) = mn$ **[The standard basis has mn vectors.]**

EXAMPLE: Dimension of Span(*S*)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ then every vector in span*(S)* is expressible as a linear combination of the vectors in *S*. Thus, if the vectors in *S* are *linearly independent*, they automatically form a basis for span*(S)*, from which we can conclude that

$\dim[\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}] = r$

In words, the dimension of the space spanned by a linearly independent set of vectors is equal to the number of vectors in that set.

EXAMPLE: Dimension of a Solution Space

Find a basis for and the dimension of the solution space of the homogeneous system

$$\begin{aligned} x_1 + 3x_2 - 2x_3 \qquad + 2x_5 \qquad &= 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\ 5x_3 + 10x_4 \qquad + 15x_6 &= 0 \\ 2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 + 18x_6 &= 0 \end{aligned}$$

Solution In Example 6 of Section 1.2 we found the solution of this system to be

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$$

which can be written in vector form as

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (-3r - 4s - 2t, r, -2s, s, t, 0)$$

or, alternatively, as

$$(x_1, x_2, x_3, x_4, x_5, x_6) = r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0)$$

This shows that the vectors

$$\mathbf{v}_1 = (-3, 1, 0, 0, 0, 0), \quad \mathbf{v}_2 = (-4, 0, -2, 1, 0, 0), \quad \mathbf{v}_3 = (-2, 0, 0, 0, 1, 0)$$

span the solution space. We leave it for you to check that these vectors are linearly independent by showing that none of them is a linear combination of the other two (but see the remark that follows). Thus, the solution space has dimension 3. ◀

Exercises**:** find a basis for the solution space of the homogeneous linear system, and find the dimension of that space.

1)

$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 \qquad + x_3 &= 0 \end{aligned}$$

2)

$$\begin{aligned} x + y + z &= 0 \\ 3x + 2y - 2z &= 0 \\ 4x + 3y - z &= 0 \\ 6x + 5y + z &= 0 \end{aligned}$$

4.5 Row Space, Column Space, and Null Space

DEFINITION: For an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

the vectors

$$\begin{aligned} \mathbf{r}_1 &= [a_{11} \quad a_{12} \quad \cdots \quad a_{1n}] \\ \mathbf{r}_2 &= [a_{21} \quad a_{22} \quad \cdots \quad a_{2n}] \\ &\vdots \\ \mathbf{r}_m &= [a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn}] \end{aligned}$$

in R^n that are formed from the rows of A are called the ***row vectors*** of A, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \quad \mathbf{c}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

in R^m formed from the columns of A are called the ***column vectors*** of A.

EXAMPLE: Row and Column Vectors of a 2 × 3 Matrix

Let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$$

The row vectors of A are

$$\mathbf{r}_1 = [2 \quad 1 \quad 0] \quad \text{and} \quad \mathbf{r}_2 = [3 \quad -1 \quad 4]$$

and the column vectors of A are

$$\mathbf{c}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{c}_3 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

DEFINITION: If A is an m × n matrix, then the subspace of R^n spanned by the row vectors of A is called the **row space** of A, and the subspace of R^m spanned by the column vectors of A is called the **column space** of A. The solution space of the homogeneous system of equations A**x** = **0**, which is a subspace of R^n, is called the **null space** of A.

THEOREM: A system of linear equations A**x** = **b** is consistent if and only if **b** is in the column space of A.

EXAMPLE: A Vector b in the Column Space of *A*

Let *A***x** = **b** be the linear system

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Show that **b** is in the column space of A by expressing it as a linear combination of the column vectors of A.

Solution Solving the system by Gaussian elimination yields (verify)

$$x_1 = 2, \quad x_2 = -1, \quad x_3 = 3$$

It follows from this and Formula (2) that

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$ ◀

The general solution of a consistent linear system can be expressed as the sum of a particular solution of that system and the general solution of the corresponding homogeneous system.

EXAMPLE: General Solution of a Linear System *A*x = b

In the concluding subsection of Section 3.4 we compared solutions of the linear systems

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 6 \end{bmatrix}$$

and deduced that the general solution **x** of the nonhomogeneous system and the general solution $\mathbf{x}_h$ of the corresponding homogeneous system (when written in column-vector form) are related by

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} -3r - 4s - 2t \\ r \\ -2s \\ s \\ t \\ \frac{1}{3} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}}_{\mathbf{x}_0} + \underbrace{r\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{x}_h}$$

EXAMPLE: Finding a Basis for the Null Space of a Matrix

Find a basis for the null space of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

Solution The null space of A is the solution space of the homogeneous linear system

A**x** = **0**, which, as shown in Example, has the basis

$$\mathbf{v}_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

THEOREM: If a matrix R is in row echelon form, then the row vectors with the leading 1's (the nonzero row vectors) form a basis for the row space of R, and the column vectors with the leading 1's of the row vectors form a basis for the column space of R.

EXAMPLE: Bases for the Row and Column Spaces of a Matrix in Row Echelon Form

Find bases for the row and column spaces of the matrix

$$R = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution Since the matrix *R* is in row echelon form, it follows from Theorem that the vectors

$$\begin{aligned} \mathbf{r}_1 &= [1 \quad -2 \quad 5 \quad 0 \quad 3] \\ \mathbf{r}_2 &= [0 \quad 1 \quad 3 \quad 0 \quad 0] \\ \mathbf{r}_3 &= [0 \quad 0 \quad 0 \quad 1 \quad 0] \end{aligned}$$

form a basis for the row space of R, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of R.

EXAMPLE: Basis for a Row Space by Row Reduction

Find a basis for the row space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Solution Since elementary row operations do not change the row space of a matrix, we can find a basis for the row space of A by finding a basis for the row space of any row echelon form of A. Reducing A to row echelon form, we obtain (verify)

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem, the nonzero row vectors of *R* form a basis for the row space of *R* and hence form a basis for the row space of *A*. These basis vectors are

$$\mathbf{r}_1 = [1 \quad -3 \quad 4 \quad -2 \quad 5 \quad 4]$$
$$\mathbf{r}_2 = [0 \quad 0 \quad 1 \quad 3 \quad -2 \quad -6]$$
$$\mathbf{r}_3 = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 5]$$

THEOREM: If A and B are row equivalent matrices, then:

(a) A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.

(b) A given set of column vectors of A forms a basis for the column space of A if and only if the corresponding column vectors of B form a basis for the column space of B.

EXAMPLE: Basis for a Column Space by Row Reduction

Find a basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

that consists of column vectors of A.

Solution We observed in Example 6 that the matrix

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the first, third, and fifth columns of R contain the leading 1's of the row vectors, the vectors

$$\mathbf{c}_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_3' = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_5' = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of R. Thus, the corresponding column vectors of A, which are

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{c}_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \quad \mathbf{c}_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

form a basis for the column space of A.

The following vectors span a subspace of R^4. Find a subset of these vectors that forms a basis of this subspace.

$$\mathbf{v}_1 = (1, 2, 2, -1), \quad \mathbf{v}_2 = (-3, -6, -6, 3),$$
$$\mathbf{v}_3 = (4, 9, 9, -4), \quad \mathbf{v}_4 = (-2, -1, -1, 2),$$
$$\mathbf{v}_5 = (5, 8, 9, -5), \quad \mathbf{v}_6 = (4, 2, 7, -4)$$

Solution If we rewrite these vectors in column form and construct the matrix that has

those vectors as its successive columns, then we obtain the matrix A in Example Thus,

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\} = \operatorname{col}(A)$$

Proceeding as in that example (and adjusting the notation appropriately), we see that the vectors $\mathbf{v}_1$, $\mathbf{v}_3$, and $\mathbf{v}_5$ form a basis for

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\} \blacktriangleleft$$

EXAMPLE: Basis for the Row Space of a Matrix

Find a basis for the row space of

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

consisting entirely of row vectors from A.

Solution We will transpose A, thereby converting the row space of A into the column space of A^T; then we will use the method of Example 7 to find a basis for the column space of A^T; and then we will transpose again to convert column vectors back to row vectors.

Transposing A yields

$$A^T = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -2 & -5 & 5 & 6 \\ 0 & -3 & 15 & 18 \\ 0 & -2 & 10 & 8 \\ 3 & 6 & 0 & 6 \end{bmatrix}$$

and then reducing this matrix to row echelon form we obtain

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first, second, and fourth columns contain the leading 1's, so the corresponding column vectors in A^T form a basis for the column space of A^T; these are

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 2 \\ -5 \\ -3 \\ -2 \\ 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{c}_4 = \begin{bmatrix} 2 \\ 6 \\ 18 \\ 8 \\ 6 \end{bmatrix}$$

Transposing again and adjusting the notation appropriately yields the basis vectors

$$\mathbf{r}_1 = [1 \quad -2 \quad 0 \quad 0 \quad 3], \quad \mathbf{r}_2 = [2 \quad -5 \quad -3 \quad -2 \quad 6],$$

$$\mathbf{r}_4 = [2 \quad 6 \quad 18 \quad 8 \quad 6]$$

for the row space of A. ◀

4.6 Rank and Nullity

DEFINITION: The common dimension of the row space and column space of a matrix *A* is called the ***rank*** of *A* and is denoted by rank*(A)*; the dimension of the null space of *A* is called the ***nullity*** of *A* and is denoted by nullity*(A)*.

EXAMPLE: Rank and Nullity of a 4 × 6 Matrix

Find the rank and nullity of the matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Solution The reduced row echelon form of A is

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(verify). Since this matrix has two leading 1's, its row and column spaces are two-dimensional and rank$(A) = 2$. To find the nullity of A, we must find the dimension of the solution space of the linear system $A\mathbf{x} = \mathbf{0}$. This system can be solved by reducing its augmented matrix to reduced row echelon form. The resulting matrix will be identical to (1), except that it will have an additional last column of zeros, and hence the corresponding system of equations will be

$$\begin{aligned} x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 &= 0 \\ x_2 - 2x_3 - 12x_4 - 16x_5 + \ \ 5x_6 &= 0 \end{aligned}$$

Solving these equations for the leading variables yields

$$\begin{aligned} x_1 &= 4x_3 + 28x_4 + 37x_5 - 13x_6 \\ x_2 &= 2x_3 + 12x_4 + 16x_5 - \ \ 5x_6 \end{aligned}$$

from which we obtain the general solution

$$\begin{aligned} x_1 &= 4r + 28s + 37t - 13u \\ x_2 &= 2r + 12s + 16t - 5u \\ x_3 &= r \\ x_4 &= s \\ x_5 &= t \\ x_6 &= u \end{aligned}$$

or in column vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Because the four vectors on the right side form a basis for the solution space, nullity*(A)* = 4.

EXAMPLE: Maximum Value for Rank

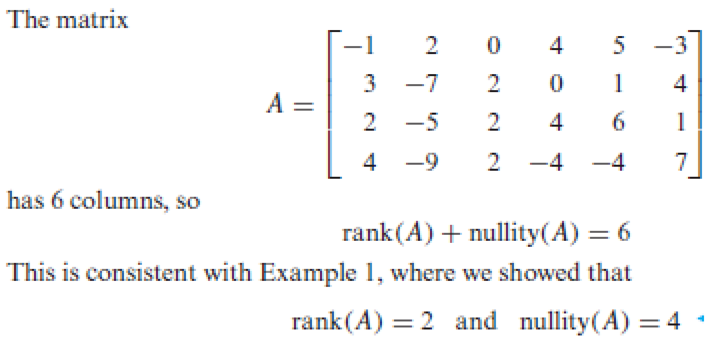
What is the maximum possible rank of an $m \times n$ matrix A that is not square?

rank$(A) \leq$ min(m, n)

in which min(m, n) is the minimum of m and n.

THEOREM: If A is a matrix with n columns, then

rank(A) + nullity(A) = n



The matrix

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

has 6 columns, so

$$\text{rank}(A) + \text{nullity}(A) = 6$$

This is consistent with Example 1, where we showed that

$$\text{rank}(A) = 2 \quad \text{and} \quad \text{nullity}(A) = 4$$

THEOREM: If A is an m × n matrix, then

(a) rank(A) = the number of leading variables in the general solution of A**x** = **0**.

(b) nullity(A) = the number of parameters in the general solution of A**x** = **0**.

EXAMPLE:

(a) Find the number of parameters in the general solution of *A***x** = **0** if *A* is a 5 × 7 matrix of rank 3.

(b) Find the rank of a 5 × 7 matrix *A* for which *A***x** = **0** has a two-dimensional solution space.

Solution (a)

nullity(*A*) = *n* − rank*(A)* = 7 − 3 = 4

Thus, there are four parameters.

***Solution* (*b*)** The matrix *A* has nullity 2, so

rank*(A)* = *n* − nullity*(A)* = 7 − 2 = 5

THEOREM: If A is any matrix, then rank(A) = rank(A^T).

If rank(A) = r, then

dim[row(A)] = r dim[col(A)] = r

dim[null(A)] = n – r dim[null(A^T)] = m – r

CHAPTER 5
Inner Product Spaces

5.1 Inner Products

DEFINITION: An **inner product** on a real vector space V is a function that associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ with each pair of vectors in V in such a way that the following axioms are satisfied for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars k.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ **[Symmetry axiom]**
2. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$ **[Additivity axiom]**
3. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$ **[Homogeneity axiom]**
4. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$ **[Positivity axiom]**

A real vector space with an inner product is called a ***real inner product space***.

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \quad \text{and} \quad d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})}$$

DEFINITION: If *V* is a real inner product space, then the ***norm*** (or ***length***) of a vector $\mathbf{v}$ in V is denoted by $\|\mathbf{v}\|$ and is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

and the ***distance*** between two vectors is denoted by $d(\mathbf{u}, \mathbf{v})$ and is defined by

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle}$$

A vector of norm 1 is called a ***unit vector***.

THEOREM: *If* **u** *and* **v** *are vectors in a real inner product space V, and if k is a scalar, then*:

(*a*) $\|\mathbf{v}\| \geq 0$ *with equality if and only if* $\mathbf{v} = \mathbf{0}$.

(*b*) $\|k\mathbf{v}\| = |k|\|\mathbf{v}\|$.

(*c*) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$.

(*d*) $d(\mathbf{u}, \mathbf{v}) \geq 0$ *with equality if and only if* $\mathbf{u} = \mathbf{v}$.

Although the Euclidean inner product is the most important inner product on R^n, there are various applications in which it is desirable to modify it by *weighting* each term differently. More precisely, if

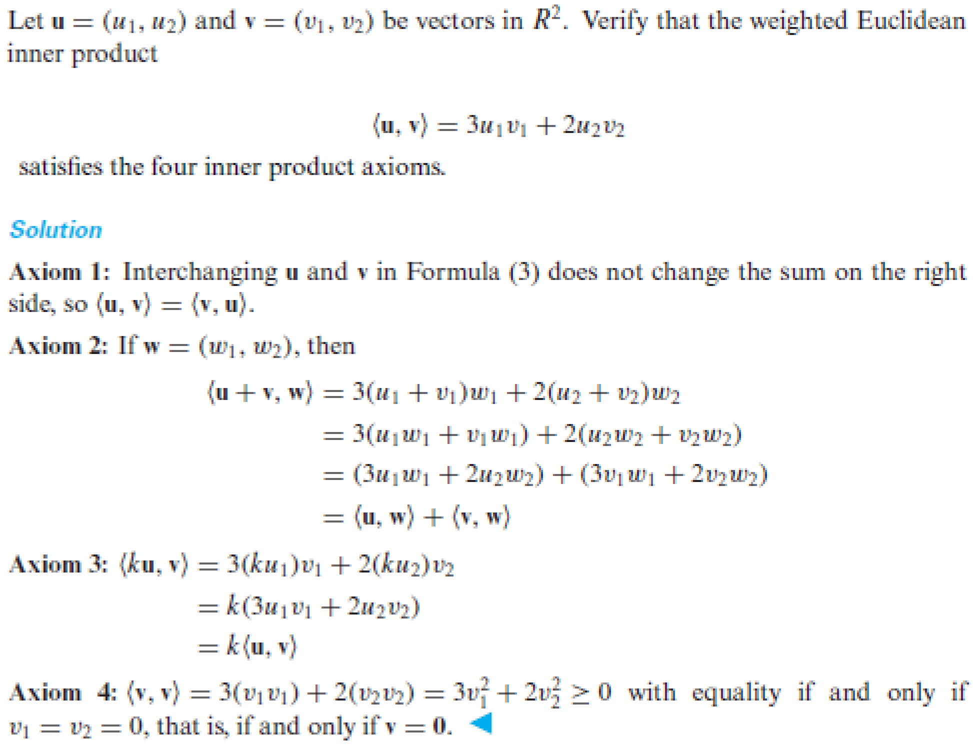
$$w_1, w_2, \ldots, w_n$$

are *positive* real numbers, which we will call ***weights***, and if **u** = $(u_1, u_2, \ldots, u_n)$ and **v** = $(v_1, v_2, \ldots, v_n)$ are vectors in R^n, then it can be shown that the formula

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1u_1v_1 + w_2u_2v_2 + \cdots + w_nu_nv_n$$

defines an inner product on *Rn* that we call the ***weighted Euclidean inner product with weights*** $w_1, w_2, \ldots, w_n$.

EXAMPLE:



Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be vectors in R^2. Verify that the weighted Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2$$

satisfies the four inner product axioms.

Solution

Axiom 1: Interchanging **u** and **v** in Formula (3) does not change the sum on the right side, so $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$.

Axiom 2: If $\mathbf{w} = (w_1, w_2)$, then

$$\begin{aligned} \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle &= 3(u_1 + v_1)w_1 + 2(u_2 + v_2)w_2 \\ &= 3(u_1w_1 + v_1w_1) + 2(u_2w_2 + v_2w_2) \\ &= (3u_1w_1 + 2u_2w_2) + (3v_1w_1 + 2v_2w_2) \\ &= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle \end{aligned}$$

Axiom 3:
$$\begin{aligned} \langle k\mathbf{u}, \mathbf{v} \rangle &= 3(ku_1)v_1 + 2(ku_2)v_2 \\ &= k(3u_1v_1 + 2u_2v_2) \\ &= k\langle \mathbf{u}, \mathbf{v} \rangle \end{aligned}$$

Axiom 4: $\langle \mathbf{v}, \mathbf{v} \rangle = 3(v_1v_1) + 2(v_2v_2) = 3v_1^2 + 2v_2^2 \geq 0$ with equality if and only if $v_1 = v_2 = 0$, that is, if and only if $\mathbf{v} = \mathbf{0}$. ◀

DEFINITION: If *V* is an inner product space, then the set of points in *V* that satisfy

$$\|u\| = 1$$

is called the ***unit sphere*** or sometimes the ***unit circle*** in *V* .

C H A P T E R 5

Eigenvalues and Eigenvectors

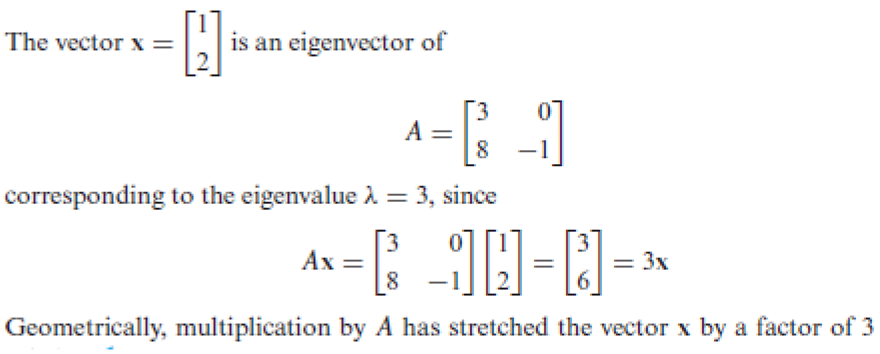
5.1 Eigenvalues and Eigenvectors

DEFINITION: If A is an n × n matrix, then a nonzero vector **x** in R^n is called an **eigenvector** of A (or of the matrix operator T_A) if A**x** is a scalar multiple of **x**; that is,

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ. The scalar λ is called an **eigenvalue** of A (or of T_A), and **x** is said to be an **eigenvector corresponding to λ**.

Example:



The vector $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of

$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

corresponding to the eigenvalue $\lambda = 3$, since

$$A\mathbf{x} = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3\mathbf{x}$$

Geometrically, multiplication by A has stretched the vector $\mathbf{x}$ by a factor of 3

THEOREM: If A is an n × n matrix, then λ is an eigenvalue of A if and only if it satisfies the equation

$$\det(\lambda I - A) = 0$$

This is called the **characteristic equation** of A.

Example: Use the characteristic equation to find all eigenvalues of this matrix.

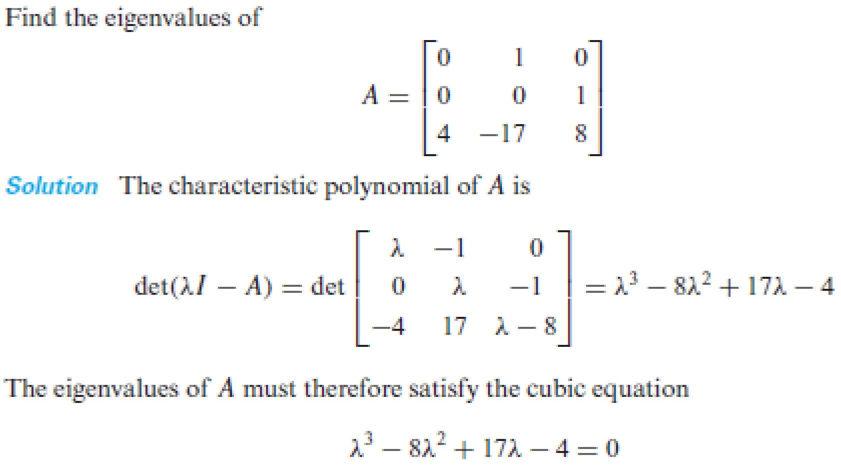
$$A = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$$

det(*λI − A)* = 0

$$\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} = \begin{vmatrix} 3-\lambda & 0 \\ -8 & \lambda+1 \end{vmatrix} = (3-\lambda)(\lambda+1) = 0$$

$\lambda = 3, -1$

Example:



Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

Solution The characteristic polynomial of A is

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{bmatrix} = \lambda^3 - 8\lambda^2 + 17\lambda - 4$$

The eigenvalues of A must therefore satisfy the cubic equation

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

$$(\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0$$

$$\lambda = 4, \quad \lambda = 2 + \sqrt{3}, \quad \text{and} \quad \lambda = 2 - \sqrt{3}$$

THEOREM: If A is an n × n triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A.

By inspection, the eigenvalues of the lower triangular matrix

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{2}{3} & 0 \\ 5 & -8 & -\frac{1}{4} \end{bmatrix}$$

are $\lambda = \frac{1}{2}$, $\lambda = \frac{2}{3}$, and $\lambda = -\frac{1}{4}$. ◀

THEOREM: If A is an n × n matrix, the following statements are equivalent.

(a) λ is an eigenvalue of A.

(b) λ is a solution of the characteristic equation det(λI − A) = 0.

(c) The system of equations (λI − A)**x** = **0** has nontrivial solutions.

(d) There is a nonzero vector **x** such that A**x** = λ**x**.

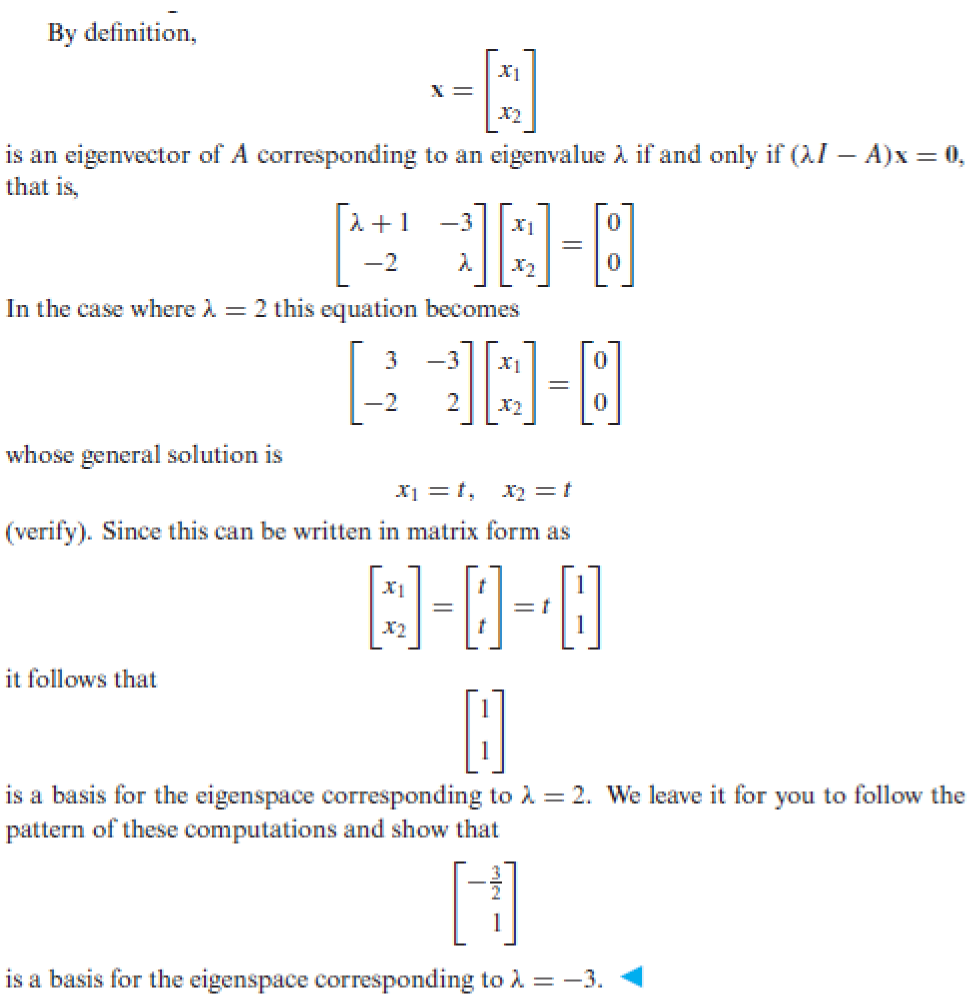
EXAMPLE: Find bases for the eigenspaces of the matrix

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$

Solution The characteristic equation of A is

$$\begin{vmatrix} \lambda + 1 & -3 \\ -2 & \lambda \end{vmatrix} = \lambda(\lambda + 1) - 6 = (\lambda - 2)(\lambda + 3) = 0$$

so the eigenvalues of A are $\lambda = 2$ and $\lambda = -3$. Thus, there are two eigenspaces of A, one for each eigenvalue.



By definition,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is an eigenvector of A corresponding to an eigenvalue λ if and only if $(\lambda I - A)\mathbf{x} = \mathbf{0}$, that is,

$$\begin{bmatrix} \lambda+1 & -3 \\ -2 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In the case where $\lambda = 2$ this equation becomes

$$\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

whose general solution is

$$x_1 = t, \quad x_2 = t$$

(verify). Since this can be written in matrix form as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

it follows that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = 2$. We leave it for you to follow the pattern of these computations and show that

$$\begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = -3$. ◀

$$\begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x_1-3x_2=0$, $-2x_1=3x_2$, $x_1=-(3\backslash 2)x_2$

$x_2=s$, $-(3/2)s$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}s \\ s \end{bmatrix} = s\begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$$

EXAMPLE: Find bases for the eigenspaces of

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$(\lambda - 1)(\lambda - 2)^2 = 0$$
$$\lambda = 1, \lambda = 2$$

is an eigenvector of A corresponding to λ if and only if **x** is a nontrivial solution of

$(\lambda I - A)\mathbf{x} = \mathbf{0}$, or in matrix form,

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In the case where $\lambda = 2$,

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system using Gaussian elimination yields (verify)

$x_1 = -s, x_2 = t, x_3 = s$

Thus, the eigenvectors of A corresponding to $\lambda = 2$ are the nonzero vectors of the form

$$x = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent ,these vectors form a basis for the eigenspace corresponding to $\lambda = 2$.

If $\lambda = 1$, then

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving this system yields

$x_1 = -2s,\ x_2 = s,\ x_3 = s$

Thus, the eigenvectors corresponding to $\lambda = 1$ are the nonzero vectors of the form

$$x = \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad \text{so that} \quad \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

is a basis for the eigenspace corresponding to $\lambda = 1$.

THEOREM: A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

EXAMPLE: Eigenvalues and Invertibility

The matrix *A* in previous Example is invertible since it has eigenvalues $\lambda = 1$ and $\lambda = 2$, neither of which is zero. We leave it for you to check this conclusion by showing that det$(A) \neq 0$.

THEOREM: Equivalent Statements

If A is an n × n matrix, then the following statements are equivalent.

(*a*) *A is invertible.*

(*b*) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(*c*) *The reduced row echelon form of A is In.*

(*d*) *A is expressible as a product of elementary matrices.*

(*e*) $A\mathbf{x} = \mathbf{b}$ *is consistent for every* $n \times 1$ *matrix* **b**.

(*f*) $A\mathbf{x} = \mathbf{b}$ *has exactly one solution for every* $n \times 1$ *matrix* **b**.

(*g*) det$(A) \neq 0$.

(*h*) *The column vectors of A are linearly independent.*

(*i*) *The row vectors of A are linearly independent.*

(*j*) *The column vectors of A span* R^n.

(*k*) *The row vectors of A span* R^n.

(*l*) *The column vectors of A form a basis for* R^n.

(*m*) *The row vectors of A form a basis for* R^n.

(*n*) *A has rank n.*

(*o*) *A has nullity* 0*.*

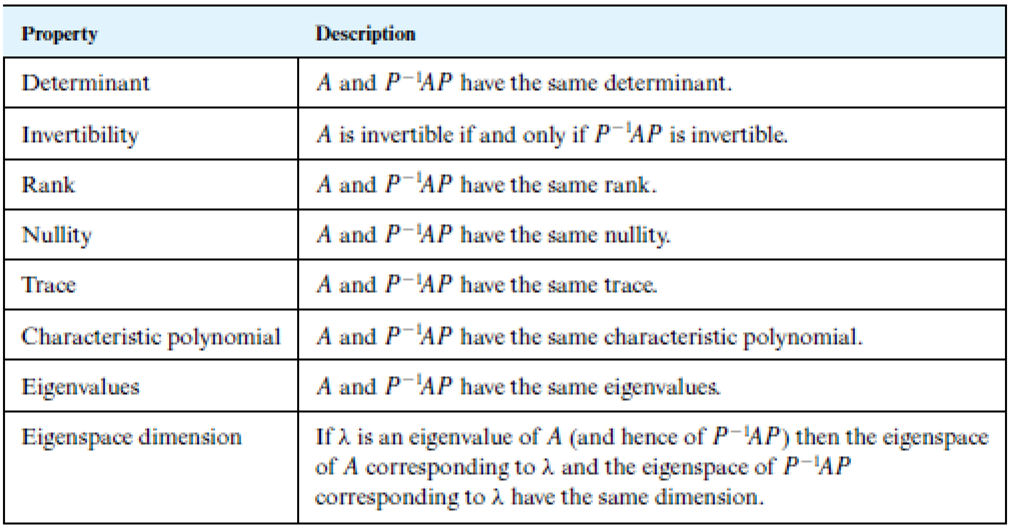
(*p*) *The orthogonal complement of the null space of A is* R^n*.*

(*q*) *The orthogonal complement of the row space of A is* {**0**}*.*

(*u*) $\lambda = 0$ *is not an eigenvalue of A.*

5.2 Diagonalization

Products of the form $P^{-1}AP$ in which A and P are n × n matrices and P is invertible will be our main topic of study in this section. There are various ways to think about such products, one of which is to view them as transformations $A \rightarrow P^{-1}AP$ in which the matrix A is mapped into the matrix $P^{-1}AP$. These are called **similarity transformations**. Such transformations are important because they preserve many properties of the matrix A. For example, if we let $B = P^{-1}AP$, then A and B have the same determinant.



Property	**Description**
Determinant	A and $P^{-1}AP$ have the same determinant.
Invertibility	A is invertible if and only if $P^{-1}AP$ is invertible.
Rank	A and $P^{-1}AP$ have the same rank.
Nullity	A and $P^{-1}AP$ have the same nullity.
Trace	A and $P^{-1}AP$ have the same trace.
Characteristic polynomial	A and $P^{-1}AP$ have the same characteristic polynomial.
Eigenvalues	A and $P^{-1}AP$ have the same eigenvalues.
Eigenspace dimension	If λ is an eigenvalue of A (and hence of $P^{-1}AP$) then the eigenspace of A corresponding to λ and the eigenspace of $P^{-1}AP$ corresponding to λ have the same dimension.

DEFINITION: If *A* and *B* are square matrices, then we say that ***B is similar to A*** if there is an invertible matrix *P* such that $B = P^{-1}AP$.

DEFINITION: A square matrix A is said to be **diagonalizable** if it is similar to some diagonal matrix; that is, if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal. In this case the matrix P is said to **diagonalize** A.

THEOREM: *If A is an n × n matrix, the following statements are equivalent.*

(*a*) *A is diagonalizable.*

(*b*) *A has n linearly independent eigenvectors.*

THEOREM

(*a*) *If* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *are distinct eigenvalues of a matrix A, and if* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ *are corresponding eigenvectors, then* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *is a linearly independent set.*

(*b*) *An n × n matrix with n distinct eigenvalues is diagonalizable.*

A Procedure for Diagonalizing an $n \times n$ Matrix

Step 1. Determine first whether the matrix is actually diagonalizable by searching for n linearly independent eigenvectors. One way to do this is to find a basis for each eigenspace and count the total number of vectors obtained. If there is a total of n vectors, then the matrix is diagonalizable, and if the total is less than n, then it is not.

Step 2. If you ascertained that the matrix is diagonalizable, then form the matrix $P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \cdots \quad \mathbf{p}_n]$ whose column vectors are the n basis vectors you obtained in Step 1.

Step 3. $P^{-1}AP$ will be a diagonal matrix whose successive diagonal entries are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ that correspond to the successive columns of P.

EXAMPLE:

Find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution In previous Example of the preceding section we found the characteristic equation of A to be

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

and we found the following bases for the eigenspaces:

$$\lambda = 2: \quad \mathbf{p}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \qquad \lambda = 1: \quad \mathbf{p}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

There are three basis vectors in total, so the matrix

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

diagonalizes A. As a check, you should verify that

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ ◀

In general, there is no preferred order for the columns of P. Since the ith diagonal entry of $P^{-1}AP$ is an eigenvalue for the ith column vector of P, changing the order of the columns of P just changes the order of the eigenvalues on the diagonal of $P^{-1}AP$. Thus, had we written

$$P = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

in the preceding example, we would have obtained

$$P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Example

Show that the following matrix is not diagonalizable:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

Solution The characteristic polynomial of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ -1 & \lambda - 2 & 0 \\ 3 & -5 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2)^2$$

so the characteristic equation is

$$(\lambda - 1)(\lambda - 2)^2 = 0$$

and the distinct eigenvalues of A are $\lambda = 1$ and $\lambda = 2$. We leave it for you to show that bases for the eigenspaces are

$$\lambda = 1: \quad \mathbf{p}_1 = \begin{bmatrix} \frac{1}{8} \\ -\frac{1}{8} \\ 1 \end{bmatrix}; \qquad \lambda = 2: \quad \mathbf{p}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Since A is a 3 × 3 matrix and there are only two basis vectors in total, A is not diagonalizable.

THEOREM: If k is a positive integer, λ is an eigenvalue of a matrix A, and **x** is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and **x** is a corresponding eigenvector.

In Example 2 we found the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

Do the same for A^7.

Solution We know from Example 2 that the eigenvalues of A are $\lambda = 1$ and $\lambda = 2$, so the eigenvalues of A^7 are $\lambda = 1^7 = 1$ and $\lambda = 2^7 = 128$. The eigenvectors $\mathbf{p}_1$ and $\mathbf{p}_2$ obtained in Example 1 corresponding to the eigenvalues $\lambda = 1$ and $\lambda = 2$ of A are also the eigenvectors corresponding to the eigenvalues $\lambda = 1$ and $\lambda = 128$ of A^7.

Suppose that A is a diagonalizable $n \times n$ matrix, that P diagonalizes A, and that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = D$$

$$P^{-1}A^kP = D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

which we can rewrite as

$$A^k = PD^kP^{-1} = P \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} P^{-1}$$

EXAMPLE: Powers of a Matrix

Find A^{13}, where

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Solution We showed in Example 1 that the matrix A is diagonalized by

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and that

$$D = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{13} = PD^{13}P^{-1} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 1^{13} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -8190 & 0 & -16382 \\ 8191 & 8192 & 8191 \\ 8191 & 0 & 16383 \end{bmatrix} \blacktriangleleft$$

CHAPTER 6

6.1 General Linear Transformations

DEFINITION If *f* is a function with domain R^n and codomain R^m, then we say that *f* is a ***transformation*** from R^n to R^m or that *f* ***maps*** from R^n to R^m, which we denote by writing $f : R^n \to R^m$

In the special case where $m = n$, a transformation is sometimes called an ***operator*** on R^n.

EXAMPLE:

The transformation from R^4 to R^3 defined by the equations

$$\begin{aligned} w_1 &= 2x_1 - 3x_2 + x_3 - 5x_4 \\ w_2 &= 4x_1 + x_2 - 2x_3 + x_4 \\ w_3 &= 5x_1 - x_2 + 4x_3 \end{aligned}$$

can be expressed in matrix form as

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

from which we see that the transformation can be interpreted as multiplication by

$$A = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix}$$

Although the image under the transformation T_A of any vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

For example, if

$$\mathbf{x} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}$$

then it follows that

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}$$

THEOREM *For every matrix A the matrix transformation* $T_A: R^n \rightarrow R^m$ *has the following properties for all vectors* **u** *and* **v** *and for every scalar* k:

(a) $T_A(\mathbf{0}) = \mathbf{0}$

(b) $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ **[Homogeneity property]**

(c) $T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$ **[Additivity property]**

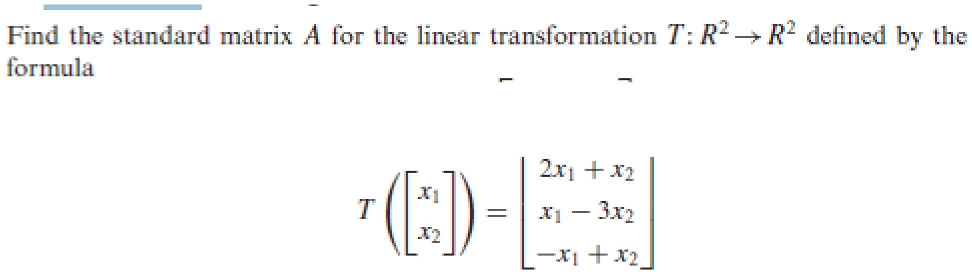
(d) $T_A(\mathbf{u} - \mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$

THEOREM $T: R^n \rightarrow R^m$ *is a matrix transformation if and only if the following relationships hold for all vectors* **u** *and* **v** *in* R^n *and for every scalar* k:

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ **[Additivity property]**

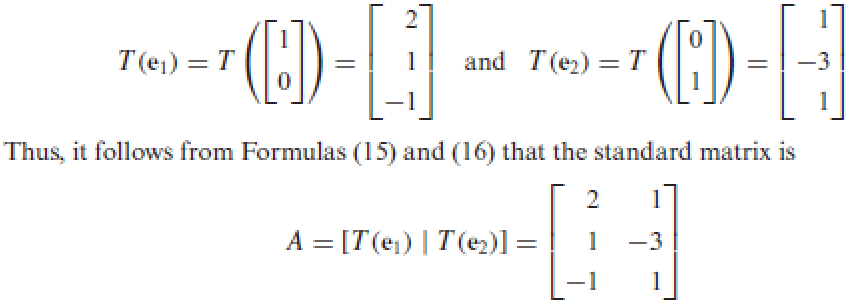
(ii) $T(k\mathbf{u}) = kT(\mathbf{u})$ **[Homogeneity property]**

EXAMPLE



Find the standard matrix A for the linear transformation $T\colon R^2 \rightarrow R^2$ defined by the formula

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix}$$



$$T(\mathrm{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad T(\mathrm{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

Thus, it follows from Formulas (15) and (16) that the standard matrix is

$$A = [T(\mathrm{e}_1) \mid T(\mathrm{e}_2)] = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}$$

For the linear transformation in Example, use the standard matrix *A* obtained in that example to find

$$T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$$

$$T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 3 \end{bmatrix}$$

EXAMPLE

Rewrite the transformation $T(x_1, x_2) = (3x_1 + x_2, 2x_1 - 4x_2)$ in column-vector form and find its standard matrix.

Solution

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_2 \\ 2x_1 - 4x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Thus, the standard matrix is

$$\begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} ◀$$

DEFINITION If *T* : *V* →*W* is a mapping from a vector space *V* to a vector space *W*, then *T* is called a ***linear transformation*** from *V* to *W* if the following two properties hold for all vectors **u** and **v** in *V* and for all scalars *k*:

(i) *T(k***u***)* = *kT(***u***)* **[Homogeneity property]**

(ii) *T(***u** + **v***)* = *T(***u***)* + *T(***v***)* **[Additivity property]**

In the special case where *V* = *W*, the linear transformation *T* is called a ***linear operator*** on the vector space *V*.

THEOREM *If T* : *V* →*W is a linear transformation, then*:

(*a*) *T(***0***)* = **0**.

(*b*) *T(***u** − **v***)* = *T(***u***)* − *T(***v***) for all* **u** *and* **v** *in V* .

EXAMPLE

Let $\mathbf{p} = p(x) = c_0 + c_1x + \cdots + c_nx^n$ be a polynomial in P_n, and define the transformation $T : P_n \to P_{n+1}$ by

$$T(\mathbf{p}) = T(p(x)) = xp(x) = c_0x + c_1x^2 + \cdots + c_nx^{n+1}$$

This transformation is linear because for any scalar k and any polynomials $\mathbf{p}_1$ and $\mathbf{p}_2$ in P_n we have

$$T(k\mathbf{p}) = T(kp(x)) = x(kp(x)) = k(xp(x)) = kT(\mathbf{p})$$

and

$$\begin{aligned} T(\mathbf{p}_1 + \mathbf{p}_2) &= T(p_1(x) + p_2(x)) = x(p_1(x) + p_2(x)) \\ &= xp_1(x) + xp_2(x) = T(\mathbf{p}_1) + T(\mathbf{p}_2) \end{aligned}$$

EXAMPLE

Let $\mathbf{v}_0$ be any fixed vector in R^n, and let $T: R^n \to R$ be the transformation

$$T(\mathbf{x}) = \langle \mathbf{x} \cdot \mathbf{v}_0 \rangle$$

that maps a vector $\mathbf{x}$ to its dot product with $\mathbf{v}_0$. This transformation is linear, for if k is any scalar, and if $\mathbf{u}$ and $\mathbf{v}$ are any vectors in R^n, then it follows from properties of the dot product in Theorem 3.2.2 that

$$T(k\mathbf{u}) = (k\mathbf{u}) \cdot \mathbf{v}_0 = k(\mathbf{u} \cdot \mathbf{v}_0) = kT(\mathbf{u})$$
$$T(\mathbf{u} + \mathbf{v}) = (\mathbf{u} + \mathbf{v}) \cdot \mathbf{v}_0 = (\mathbf{u} \cdot \mathbf{v}_0) + (\mathbf{v} \cdot \mathbf{v}_0) = T(\mathbf{u}) + T(\mathbf{v})$$

EXAMPLE

Let M_{nn} be the vector space of $n \times n$ matrices. In each part determine whether the transformation is linear.

(a) $T_1(A) = A^T$ (b) $T_2(A) = \det(A)$

Solution (a) It follows from parts (*b*) and (*d*) of Theorem 1.4.8 that

$$T_1(kA) = (kA)^T = kA^T = kT_1(A)$$
$$T_1(A + B) = (A + B)^T = A^T + B^T = T_1(A) + T_1(B)$$

so T_1 is linear.

Solution (b) It follows from Formula (1) of Section 2.3 that

$$T_2(kA) = \det(kA) = k^n \det(A) = k^n T_2(A)$$

Thus, T_2 is not homogeneous and hence not linear if $n > 1$. Note that additivity also fails because we showed in Example 1 of Section 2.3 that $\det(A + B)$ and $\det(A) + \det(B)$ are not generally equal.

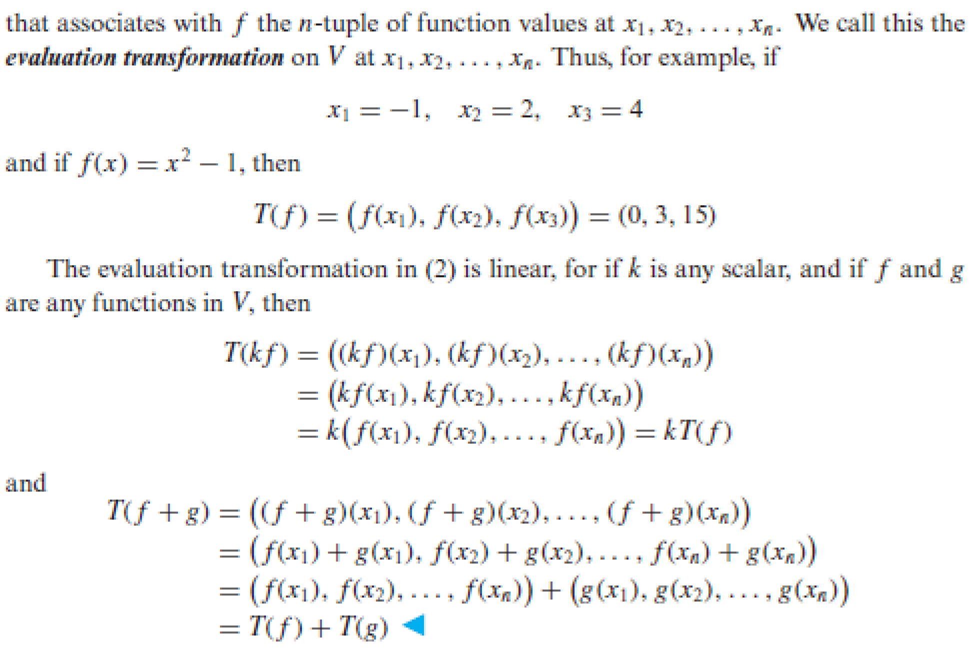
EXAMPLE

Let V be a subspace of $F(-\infty, \infty)$, let

$$x_1, x_2, \ldots, x_n$$

be a sequence of distinct real numbers, and let $T: V \to R^n$ be the transformation

$$T(f) = \big(f(x_1), f(x_2), \ldots, f(x_n)\big)$$



that associates with f the n-tuple of function values at $x_1, x_2, \dots, x_n$. We call this the ***evaluation transformation*** on V at $x_1, x_2, \dots, x_n$. Thus, for example, if

$$x_1 = -1, \quad x_2 = 2, \quad x_3 = 4$$

and if $f(x) = x^2 - 1$, then

$$T(f) = \big(f(x_1), f(x_2), f(x_3)\big) = (0, 3, 15)$$

The evaluation transformation in (2) is linear, for if k is any scalar, and if f and g are any functions in V, then

$$\begin{aligned} T(kf) &= \big((kf)(x_1), (kf)(x_2), \dots, (kf)(x_n)\big) \\ &= \big(kf(x_1), kf(x_2), \dots, kf(x_n)\big) \\ &= k\big(f(x_1), f(x_2), \dots, f(x_n)\big) = kT(f) \end{aligned}$$

and

$$\begin{aligned} T(f+g) &= \big((f+g)(x_1), (f+g)(x_2), \dots, (f+g)(x_n)\big) \\ &= \big(f(x_1)+g(x_1), f(x_2)+g(x_2), \dots, f(x_n)+g(x_n)\big) \\ &= \big(f(x_1), f(x_2), \dots, f(x_n)\big) + \big(g(x_1), g(x_2), \dots, g(x_n)\big) \\ &= T(f) + T(g) \end{aligned}$$ ◀

THEOREM *Let $T : V \rightarrow W$ be a linear transformation, where V is finite-dimensional.*

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V, then the image of any vector $\mathbf{v}$ in V can

be expressed as

$$T(\mathbf{v}) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_nT(\mathbf{v}_n)$$

where $c_1, c_2, \dots, c_n$ are the coefficients required to express $\mathbf{v}$ as a linear combination

of the vectors in the basis S.

EXAMPLE

Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for R^3, where

$$\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = (1, 1, 0), \quad \mathbf{v}_3 = (1, 0, 0)$$

Let $T: R^3 \to R^2$ be the linear transformation for which

$$T(\mathbf{v}_1) = (1, 0), \quad T(\mathbf{v}_2) = (2, -1), \quad T(\mathbf{v}_3) = (4, 3)$$

Find a formula for $T(x_1, x_2, x_3)$, and then use that formula to compute $T(2, -3, 5)$.

Solution We first need to express $\mathbf{x} = (x_1, x_2, x_3)$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. If we write

$$(x_1, x_2, x_3) = c_1(1, 1, 1) + c_2(1, 1, 0) + c_3(1, 0, 0)$$

then on equating corresponding components, we obtain

$$\begin{aligned} c_1 + c_2 + c_3 &= x_1 \\ c_1 + c_2 \qquad &= x_2 \\ c_1 \qquad\qquad &= x_3 \end{aligned}$$

which yields $c_1 = x_3$, $c_2 = x_2 - x_3$, $c_3 = x_1 - x_2$, so

$$\begin{aligned} (x_1, x_2, x_3) &= x_3(1, 1, 1) + (x_2 - x_3)(1, 1, 0) + (x_1 - x_2)(1, 0, 0) \\ &= x_3\mathbf{v}_1 + (x_2 - x_3)\mathbf{v}_2 + (x_1 - x_2)\mathbf{v}_3 \end{aligned}$$

Thus

$$\begin{aligned} T(x_1, x_2, x_3) &= x_3T(\mathbf{v}_1) + (x_2 - x_3)T(\mathbf{v}_2) + (x_1 - x_2)T(\mathbf{v}_3) \\ &= x_3(1, 0) + (x_2 - x_3)(2, -1) + (x_1 - x_2)(4, 3) \\ &= (4x_1 - 2x_2 - x_3, 3x_1 - 4x_2 + x_3) \end{aligned}$$

From this formula we obtain

$$T(2, -3, 5) = (9, 23)$$

EXAMPLE An Integral Transformation

Let $V = C(-\infty, \infty)$ be the vector space of continuous functions on the interval $(-\infty, \infty)$, let $W = C^1(-\infty, \infty)$ be the vector space of functions with continuous first derivatives on $(-\infty, \infty)$, and let $J: V \to W$ be the transformation that maps a function f in V into

$$J(f) = \int_0^x f(t)\,dt$$

For example, if $f(x) = x^2$, then

$$J(f) = \int_0^x t^2\,dt = \frac{t^3}{3}\bigg]_0^x = \frac{x^3}{3}$$

The transformation $J: V \to W$ is linear, for if k is any constant, and if f and g are any functions in V, then properties of the integral imply that

$$J(kf) = \int_0^x kf(t)\,dt = k\int_0^x f(t)\,dt = kJ(f)$$

$$J(f + g) = \int_0^x (f(t) + g(t))\,dt = \int_0^x f(t)\,dt + \int_0^x g(t)\,dt = J(f) + J(g) \blacktriangleleft$$

Kernel and Range

DEFINITION If $T: V \rightarrow W$ is a linear transformation, then the set of vectors in V that T maps into **0** is called the ***kernel*** of T and is denoted by ker(T). The set of all vectors in

W that are images under *T* of at least one vector in *V* is called the ***range*** of *T* and is denoted by *R(T)*.

EXAMPLE Kernel and Range of the Zero Transformation

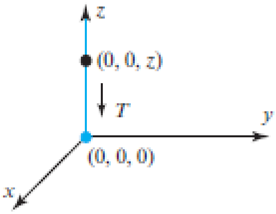
Let *T* : *V* →*W* be the zero transformation. Since *T* maps *every* vector in *V* into **0**, it follows that ker*(T)* = *V.* Moreover, since **0** is the *only* image under *T* of vectors in *V*, it follows that *R(T)* = {**0**}.

EXAMPLE Kernel and Range of the Identity Operator

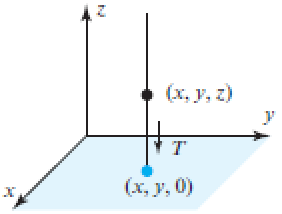
Let *I* : *V* →*V* be the identity operator. Since *I* (**v**) = **v** for all vectors in *V*, *every* vector in *V* is the image of some vector (namely, itself); thus *R(I)* = *V.* Since the *only* vector that *I* maps into **0** is **0**, it follows that ker*(I)* = {**0**}.

EXAMPLE 16 Kernel and Range of an Orthogonal Projection

Let *T* :R^3→R^3 be the orthogonal projection onto the *xy*-plane. As illustrated in Figure *a*, the points that *T* maps into **0** = *(*0*,* 0*,* 0*)* are precisely those on the *z*-axis, so ker*(T)* is the set of points of the form *(*0*,* 0*, z)*. As illustrated in Figure *b*, *T* maps the points in R^3 to the *xy*-plane, where each point in that plane is the image of each point on the vertical line above it. Thus, *R(T)* is the set of points of the form *(x, y,* 0*)*.



(*a*) ker(*T*) is the *z*-axis.



(*b*) *R*(*T*) is the entire *xy*-plane

EXAMPLE Kernel of a Differentiation Transformation

Let *V* = $C^1(-\infty, \infty)$ be the vector space of functions with continuous first derivatives on *(*−∞*,* ∞*)*, let *W* = *F(*−∞*,* ∞*)* be the vector space of all real-valued functions defined on *(*−∞*,* ∞*)*, and let *D*: *V* →*W* be the differentiation transformation *D(* **f** *)* = $f'(x)$.

The kernel of *D* is the set of functions in *V* with derivative zero. From calculus, this is the set of constant functions on *(*−∞*,* ∞*)*.

THEOREM *If $T : V \to W$ is a linear transformation, then*:

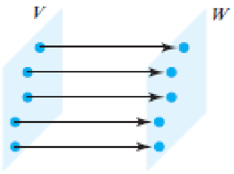
(*a*) *The kernel of T is a subspace of V.*

(*b*) *The range of T is a subspace of W.*

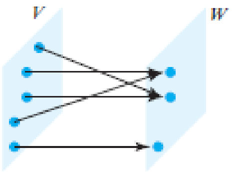
Compositions and Inverse Transformations

DEFINITION If $T : V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be ***one-to-one*** if T maps distinct vectors in V into distinct vectors in W.

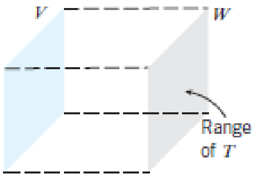
DEFINITION If $T : V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be ***onto*** (or ***onto W***) if every vector in W is the image of at least one vector in V.



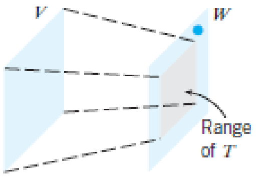
One-to-one. Distinct vectors in V have distinct images in W.



Not one-to-one. There exist distinct vectors in V with the same image.



Onto W. Every vector in W is the image of some vector in V.



Not onto W. Not every vector in W is the image of some vector in V.

THEOREM *If $T : V \to W$ is a linear transformation, then the following statements are equivalent.*

(*a*) *T is one-to-one.*

(*b*) $\ker(T) = \{\mathbf{0}\}$.

THEOREM Let $T : R^n \to R^n$ be a linear transformations and let A be the standard matrix of T. Then, the following are equivalent,

1. T is invertible.

2. T is 1-1.

3. A is invertible.

And, if T is invertible, then the standard matrix of T^{-1} is A^{-1} .

EXAMPLE An Inverse Transformation

$$[T] = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

Let $T : R^3 \to R^3$ be the linear operator defined by the formula

$$T(x_1, x_2, x_3) = (3x_1 + x_2, -2x_1 - 4x_2 + 3x_3, 5x_1 + 4x_2 - 2x_3)$$

Determine whether T is one-to-one; if so, find $T^{-1}(x_1, x_2, x_3)$.

Det([T])= (24+15+0)-(0+36+4)=-1 then T is 1-1

This matrix is invertible, the standard matrix for T^{-1} is

$$[T^{-1}] = [T]^{-1} = \begin{bmatrix} 4 & -2 & -3 \\ -11 & 6 & 9 \\ -12 & 7 & 10 \end{bmatrix}$$

It follows that

$$T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = [T^{-1}]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 & -2 & -3 \\ -11 & 6 & 9 \\ -12 & 7 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 - 2x_2 - 3x_3 \\ -11x_1 + 6x_2 + 9x_3 \\ -12x_1 + 7x_2 + 10x_3 \end{bmatrix}$$

Expressing this result in horizontal notation yields

$$T^{-1}(x_1, x_2, x_3) = (4x_1 - 2x_2 - 3x_3, -11x_1 + 6x_2 + 9x_3, -12x_1 + 7x_2 + 10x_3)$$ ◀

EXAMPLE Find the standard matrix for the linear transformation defined by the linear equations and determine whether the operation is 1-1

a) $w_1=8x_1+4x_2$

$w_2=2x_1+x_2$

b) $w_1=-x_1+3x_2+2x_3$

$w_2=2x_1+4x_3$

$w_3=x_1+3x_2+5x_3$

Solution:

a) $[T]=\begin{bmatrix} 8 & 4 \\ 2 & 1 \end{bmatrix}$

det[T]= 8-8=0 then [T] not invertible and not 1-1

b) $[T]=\begin{bmatrix} -1 & 3 & 2 \\ 2 & 0 & 4 \\ 1 & 3 & 5 \end{bmatrix}$

det[T]= -6 then [T] is invertible and 1-1