Chapter 5

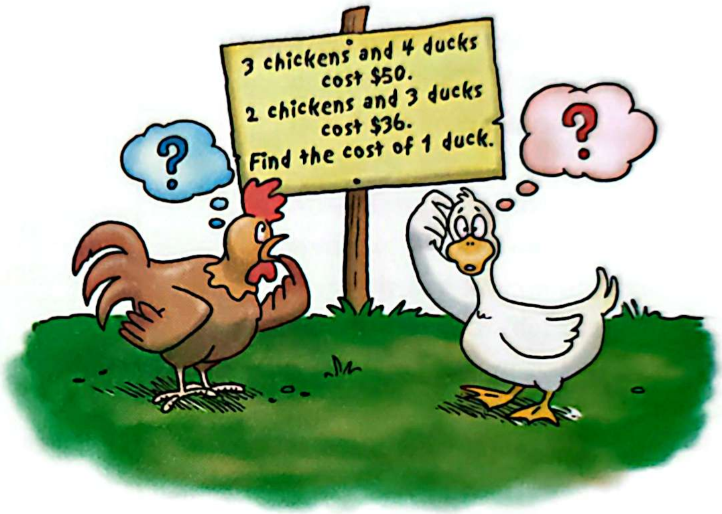
Simultaneous Linear Equations

DISCOVER!

How to:

- solve simultaneous linear equations using the substitution method
- solve simultaneous linear equations using the elimination method
- solve simultaneous linear equations using the graphical method
- solve word problems by formulating a pair of simultaneous linear equations in two unknowns

In primary school, you have learned how to solve problems of the type shown below using the model method. We can also solve problems of this type by translating the problem into a pair of simultaneous linear equations and then solving it algebraically.



The problem above, for example, can first be translated into the following pair of simultaneous linear equations and then solved algebraically:

$$3c + 4d = 50$$
$$2c + 3d = 36$$

where c is the cost of 1 chicken and d is the cost of 1 duck.

Simultaneous linear equations are used in many occupational fields to model the world around us. For example, a biologist may need to solve simultaneous linear equations to get an idea of how a population of animals may change over time. A chemical engineer working in an oil refinery may need to use simultaneous linear equations to help determine how much of each type of crude oil to use.

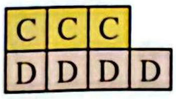
In this chapter, you will learn how to solve simultaneous linear equations using different methods as well as how to formulate simultaneous linear equations in two unknowns in order to solve word problems.

5.1 Introduction

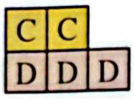
If we use the model method learned in primary school to solve the problem shown on the previous page, we have the following:

Recall problem on previous page:

3 chickens and 4 ducks cost \$50.
2 chickens and 3 ducks cost \$36.
Find the cost of 1 duck.

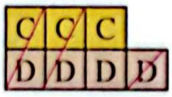
Model 1: 

= 50

Model 2: 

= 36

Using **Model 1 – Model 2**, we have:



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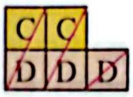


= 50 – 36

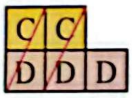
⇒ 

= 14

Model 3: ∴ 

$= 2 \times 14 = 28$

Next, using **Model 2 – Model 3**, we have:



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= 36 – 28

⇒ 

= 8

Therefore, each duck costs \$8.

We can also use algebraic methods to solve problems of the type shown above. Using these methods may be less tedious than using the model method in some cases. Let us now explore the concept of solving such problems algebraically.

Consider first the linear equation $x + y = 6$ which contains two unknowns. There are many pairs of values of x and y that will satisfy this equation as shown below.

For example, in $x + y = 6$,

$x = 5, y = 1$	gives	$5 + 1 = 6$
$x = 4, y = 2$	gives	$4 + 2 = 6$
$x = 2, y = 4$	gives	$2 + 4 = 6$
$x = -3, y = 9$	gives	$-3 + 9 = 6$
$x = 1\frac{1}{2}, y = 4\frac{1}{2}$	gives	$1\frac{1}{2} + 4\frac{1}{2} = 6$
and so on.		

Therefore, a linear equation containing two unknowns will have infinitely many pairs of solutions for x and y.

Now consider another linear equation $x - y = 4$ which also contains two unknowns. Similarly, there are many pairs of values of x and y that will satisfy this equation.

For example, in $x - y = 4$,

$x = 5, y = 1$	gives	$5 - 1 = 4$
$x = 6, y = 2$	gives	$6 - 2 = 4$
$x = 7, y = 3$	gives	$7 - 3 = 4$
$x = -3, y = -7$	gives	$-3 - (-7) = 4$
$x = -\frac{1}{2}, y = -4\frac{1}{2}$	gives	$-\frac{1}{2} - \left(-4\frac{1}{2}\right) = 4$
and so on.		

What if we want to find the values of x and y that satisfy **both** $x + y = 6$ and $x - y = 4$? When we do this, we are said to be solving **simultaneous linear equations**.

note

Doing two things 'simultaneously' means doing both of them at the same time. Thus, **solving simultaneous linear equations** means solving both equations at the same time to get a common solution.

From the above list of values of x and y, we can easily see that $x = 5$ and $y = 1$ satisfy both equations. Therefore, the solution for the simultaneous linear equations $x + y = 6$ and $x - y = 4$ is $\mathbf{x = 5}$ and $\mathbf{y = 1}$.

It is usually tedious to obtain the solution by listing possible values as what we have done above. In the following sections, we will learn about other ways of solving simultaneous linear equations using methods involving algebra and graphs.

5.2 Algebraic Method of Solving Simultaneous Linear Equations

There are two algebraic methods of solving simultaneous linear equations:

(a) substitution method
(b) elimination method

5.2.1 Substitution Method

The method of **substitution** is especially useful when at least one equation is given in the form '$y = \ldots$' or '$x = \ldots$'. We can then substitute the expression given on the right hand side directly into the other equation to obtain a linear equation in one unknown.

Example 1

Solve the simultaneous linear equations

$$y = 3x - 5,$$
$$y = 2x + 3.$$

Solution

$$y = 3x - 5 \quad \ldots\ldots\ldots\ldots \textit{(1)}$$
$$y = 2x + 3 \quad \ldots\ldots\ldots\ldots \textit{(2)}$$

Substitute y from *(1)* into *(2)*:
$y = 2x + 3$
$3x - 5 = 2x + 3$

Substitute *(1)* into *(2)*,

$$3x - 5 = 2x + 3$$
$$3x - 2x = 3 + 5$$
$$x = 8$$

We can also substitute $x = 8$ into *(2)* to get the answer for y.

Substitute $x = 8$ into *(1)*,

$$y = 3(8) - 5$$
$$= 19$$

$\therefore$ the solution is $x = 8$ and $y = 19$.

note

If we substituted $x = 8$ into *(2)* earlier, we have to use *(1)* instead to check our answers.
If LHS $\neq$ RHS, the solution is wrong. Check your working again.

Check: Substitute $x = 8$ and $y = 19$ into *(2)*,

LHS $= y = 19$
RHS $= 2x + 3$
$= 2(8) + 3$
$= 19$
$\therefore$ LHS = RHS

Example 2

Solve the simultaneous linear equations

$$2x + 3y = 4,$$
$$y = 3x + 5.$$

Solution

$$2x + 3y = 4 \quad \text{.........................} \quad (1)$$
$$y = 3x + 5 \quad \text{.........................} \quad (2)$$

Substitute *(2)* into *(1)*,

$$\begin{aligned} 2x + 3(3x + 5) &= 4 \\ 2x + 9x + 15 &= 4 \\ 11x &= 4 - 15 \\ &= -11 \\ x &= -1 \end{aligned}$$

note

It is more difficult to substitute *(1)* into *(2)* because we would first have to manipulate the equation to make x or y alone on the left hand side first.

Substitute $x = -1$ into *(2)*,

$$\begin{aligned} y &= 3(-1) + 5 \\ &= 2 \end{aligned}$$

$\therefore$ the solution is $x = -1$ and $y = 2$.

Check: Substitute $x = -1$ and $y = 2$ into *(1)*,

$$\begin{aligned} \text{LHS} &= 2x + 3y \\ &= 2(-1) + 3(2) \\ &= 4 \\ \text{RHS} &= 4 \end{aligned}$$

$\therefore$ LHS = RHS

Example 3

Solve the simultaneous linear equations

$$2x = 5y - 1,$$
$$2x = 3y + 1.$$

Solution

$$2x = 5y - 1 \quad \text{.........................} \quad (1)$$
$$2x = 3y + 1 \quad \text{.........................} \quad (2)$$

Substitute $2x$ from *(1)* into *(2)*:

$2x = 3y + 1$

$5y - 1 = 3y + 1$

Substitute *(1)* into *(2)*,

$$5y - 1 = 3y + 1$$
$$5y - 3y = 1 + 1$$
$$2y = 2$$
$$y = 1$$

Substitute $y = 1$ into *(1)*,

$$2x = 5(1) - 1$$
$$= 4$$
$$x = 2$$

Remember to check your answers.

$\therefore$ the solution is $x = 2$ and $y = 1$.

Example 4

Solve the simultaneous linear equations

$$3x + 2y = 13,$$
$$2x + 3y = 12.$$

Solution

$3x + 2y = 13$ .... *(1)*

$2x + 3y = 12$ .... *(2)*

Manipulate *(1)* to make x alone on the left hand side of the equation.

From *(1)*,

$$3x = 13 - 2y$$

$x = \dfrac{13 - 2y}{3}$ .... *(3)*

Substitute *(3)* into *(2)*,

$$2\left(\frac{13 - 2y}{3}\right) + 3y = 12$$
$$\frac{26 - 4y}{3} + 3y = 12$$
$$26 - 4y + 9y = 36$$
$$5y = 36 - 26$$
$$= 10$$
$$y = 2$$

note

Alternatively, we can substitute $y = 2$ into *(1)* or *(2)* to obtain the same answer for x.

Substitute $y = 2$ into *(3)*,

$$x = \frac{13 - 2(2)}{3}$$
$$= 3$$

We cannot check the answer by substituting the solution into *(1)* as *(1)* and *(3)* are essentially the same equation.

$\therefore$ the solution is $x = 3$ and $y = 2$.

Exercise 5A

1. Use the substitution method to solve the following pairs of simultaneous linear equations.

(a) $y = 2x + 1$
$y = x - 1$

(b) $x = 3y - 1$
$x = 2y + 1$

(c) $3x + 4y = 11$
$y = 9 - 2x$

(d) $y = 3 - 4x$
$6x - 5y = -2$

(e) $2x = 3y + 4$
$2x = 5y + 8$

(f) $3y = 2x + 1$
$3y = 3x + 4$

(g) $3x - 4y = 2$
$3x = 7y - 1$

(h) $5y - 3x = 2$
$5y = 8x - 3$

2. Use the substitution method to solve the following pairs of simultaneous linear equations.

(a) $5x + 2y = -1$
$x - 3y = -7$

(b) $3x + 7y = 4$
$2x + y = -1$

(c) $4x + 5y = 1$
$4x - y = -5$

(d) $5y + 3x = 20$
$-x + 6y = 24$

(e) $5y - 7 = 3x$
$5y - 17x = -7$

(f) $7x - 2y = 8$
$3y + 7x = 23$

3. Use the substitution method to solve the following pairs of simultaneous linear equations.

(a) $2x + 3y = -3$
$3x - 4y = 4$

(b) $5x + 4y = 5$
$7x - 3y = 50$

(c) $2x + 2y = 3$
$8x - 3y = 1$

(d) $5x = 1 + 4y$
$2y = 1 + 4x$

(e) $3x + 2y = -10$
$4x + 9y = -7$

(f) $5x - 4y = 35$
$6y + 13x = 9$

(g) $2x - 3y = 7$
$5x - 7y = 18$

(h) $2x + \frac{1}{2}y = 16$
$3x - 2y = 13$

TIME-OUT ACTIVITY

Spot the Mistakes

Anthony was given a pair of simultaneous linear equations to solve.

$y = x - 1$ .... *(1)*
$2x - 3y = 4$ .... *(2)*

His solution is shown below.

Substitute *(1)* into *(2)*,
$2x - 3(x - 1) = 4$
$2x - 3x - 3 = 4$
$-x = 4 + 3$
$x = -7$

Substitute $x = -7$ into *(1)*,
$y = -7 - 1$
$= -8$

$\therefore$ the solution is $x = -7$ and $y = -8$.

Check: Substitute $x = -7$ and $y = -8$ into *(1)*,

$$\begin{aligned} \text{LHS} &= y = -8 \\ \text{RHS} &= x - 1 \\ &= -7 - 1 \\ &= -8 \\ \therefore \quad \text{LHS} &= \text{RHS} \end{aligned}$$

Anthony was happy that he has found the solution and he has checked to ensure that his solution is correct. However, his teacher said that Anthony's solution and method of checking is wrong.

Discuss with your classmates and explain in your Mathematics Journal why Anthony's solution is wrong. How would you advise Anthony to check his solution?

5.2.2 Elimination Method

Did you know?
The elimination method was first discovered by the ancient Chinese and was included in the 8th chapter of a classic mathematics text (*Jiuzhang suanshu* or *Nine Chapters of the Mathematical Art*) dated around 100 BC.

We have seen how we can use the substitution method to solve a pair of simultaneous linear equations. In this section, we will learn about the **elimination** method. Sometimes, it is more convenient to use the elimination method to solve a pair of simultaneous linear equations as compared to using the substitution method.

To use this method, we either add or subtract the two equations to eliminate one of the unknowns and obtain an equation in only one unknown.

Example 5

Solve the simultaneous linear equations

$$2x + y = 5,$$
$$5x - y = 9.$$

Solution

$$2x + y = 5 \quad \text{.............................. } (1)$$
$$5x - y = 9 \quad \text{.............................. } (2)$$

(1) + *(2)*,

$$\begin{aligned} (2x + y) + (5x - y) &= 5 + 9 \\ 7x &= 14 \quad \leftarrow y \text{ is eliminated} \\ x &= 2 \end{aligned}$$

Adding the LHS of *(1)* and the LHS of *(2)* eliminates the terms in y, leaving only one unknown, i.e., x.

note

Use the same steps as in the substitution method to find the value of the other unknown and to check your answers.

Substitute $x = 2$ into *(1)*,

$$2(2) + y = 5$$
$$4 + y = 5$$
$$y = 5 - 4$$
$$= 1$$

$\therefore$ the solution is $x = 2$ and $y = 1$.

Check: Substitute $x = 2$ and $y = 1$ into *(2)*,

$$\text{LHS} = 5x - y$$
$$= 5(2) - 1$$
$$= 9$$
$$\text{RHS} = 9$$
$$\therefore \quad \text{LHS} = \text{RHS}$$

Example 6

Solve the simultaneous linear equations

$$x + y = 0,$$
$$x - 2y = -6.$$

Solution

$$x + y = 0 \text{ .... } (1)$$
$$x - 2y = -6 \text{ .... } (2)$$

Eliminate x by subtracting each side of *(2)* from the respective sides of *(1)*. Alternatively, we can subtract *(1)* from *(2)* to eliminate x.

(1) – *(2)*,

$$(x + y) - (x - 2y) = 0 - (-6)$$
$$x + y - x + 2y = 0 + 6$$
$$3y = 6 \leftarrow x \text{ is eliminated}$$
$$y = 2$$

Substitute $y = 2$ into *(1)*,

$$x + 2 = 0$$
$$x = 0 - 2$$
$$= -2$$

$\therefore$ the solution is $x = -2$ and $y = 2$.

Check: Substitute $x = -2$ and $y = 2$ into *(2)*,

$$\text{LHS} = x - 2y$$
$$= -2 - 2(2)$$
$$= -6$$
$$\text{RHS} = -6$$
$$\therefore \quad \text{LHS} = \text{RHS}$$

note
To be able to solve using the elimination method, we need to first ensure that at least one of the variables has the same numerical coefficient in both equations.

Example 7

Solve the simultaneous linear equations

$2x + y = -1,$
$3x - 4y = 4.$

Solution

$2x + y = -1$ .... *(1)*
$3x - 4y = 4$ .... *(2)*

We multiply *(1)* by 4 to obtain the same numerical coefficient of y (disregard the sign) as in *(2)*.

(1) × 4,
$(2x + y) \times 4 = (-1) \times 4$
$8x + 4y = -4$ ....*(3)*

Add *(2)* and *(3)* to eliminate y.

(2) + *(3)*,
$(3x - 4y) + (8x + 4y) = 4 + (-4)$
$11x = 0$
$x = 0$

Substitute $x = 0$ into *(1)*,
$2(0) + y = -1$
$y = -1$

Remember to check your answers.

∴ the solution is $x = 0$ and $y = -1$.

In the above example, you may wish to eliminate x instead of y. In this case, to obtain the same coefficient of x in both equations, we have to multiply *(1)* by 3 and *(2)* by 2 to obtain a common coefficient of 6. Which is easier to eliminate in the above example, x or y?

Test your understanding of solving simultaneous linear equations at http://www.quia.com/pp/11581.html. You will be in for a surprise!

Example 8

Solve the simultaneous linear equations

$5x - 4y = 2,$
$2x + 3y = 10.$

Solution

$5x - 4y = 2$ .... *(1)*
$2x + 3y = 10$ .... *(2)*

We multiply *(1)* by 3 and *(2)* by 4 so that the numerical coefficients of y (disregard the sign) are the same. The L.C.M. of 4 and 3 is 12.

(1) × 3,
$15x - 12y = 6$ ....*(3)*

(2) × 4,
$8x + 12y = 40$ ....*(4)*

note

When the signs of the common terms in each equation are **different**, we **add** the equations to eliminate them. When the signs are the **same**, we **subtract** the equations.

(3) + *(4)*,

$$(15x - 12y) + (8x + 12y) = 6 + 40$$
$$23x = 46$$
$$x = 2$$

Substitute $x = 2$ into *(1)*,

$$5(2) - 4y = 2$$
$$10 - 4y = 2$$
$$-4y = 2 - 10$$
$$-4y = -8$$
$$y = \frac{-8}{-4}$$
$$= 2$$

$\therefore$ the solution is $x = 2$ and $y = 2$.

Exercise 5B

1. Use the elimination method to solve the following pairs of simultaneous linear equations.

(a) $x + y = 3$, $x - y = 1$
(b) $2a + b = 8$, $3a - b = 17$
(c) $4p + 2q = 24$, $p - 2q = 1$
(d) $x + y = -1$, $3x + y = 1$
(e) $f - 2g = 5$, $3f - 2g = 3$
(f) $2c - 3d = 11$, $7c - 3d = 1$

2. Use the elimination method to solve the following pairs of simultaneous linear equations.

(a) $5x - 2y = 11$, $x + y = 5$
(b) $4p - q = -2$, $3p + 2q = -7$
(c) $4e + 3f = 9$, $e + f = 2$
(d) $2r + 7s = 26$, $r + s = 3$
(e) $2j - 3k = 3$, $3j - k = -13$
(f) $7w - v = -29$, $2w - 3v = -11$

3. Use the elimination method to solve the following pairs of simultaneous linear equations.

(a) $4a + 3b = 0$, $7a - 2b = -29$
(b) $2c - 5d = 21$, $7c + 3d = 12$
(c) $3x + 2y = 6$, $5x + 3y = 11$
(d) $5x - 4y = -1$, $2x - 3y = 1$
(e) $3x - 8y = 5$, $4x + 5y = 38$
(f) $7x + 2y = 3$, $2x - 9y = -47$

4. Use the elimination method to solve the following pairs of simultaneous linear equations.

(a) $-3c + 4d = 4$, $9c - 2d = 3$
(b) $2m + 3n = -2$, $8m - 9n = -1$
(c) $x + \frac{1}{2}y = 4$, $\frac{1}{2}x - \frac{1}{2}y = \frac{1}{2}$
(d) $\frac{x}{2} + \frac{y}{3} = 3$, $\frac{x}{4} + \frac{2y}{3} = 3$

5. Use a **suitable method** to solve the following pairs of simultaneous linear equations.

(a) $5a - 3b = 6$, $5a - 4b = 8$
(b) $c + d = \frac{1}{2}$, $c - d = \frac{1}{4}$
(c) $2e - 4f = 9$, $2e - 2f = 5$
(d) $x + 2y = 5$, $3x + 4y = 11$
(e) $3x + 4y = -14$, $x - 3y = 17$
(f) $3x + 2y = 10$, $x - y = 2.5$
(g) $3x + 2y = 5$, $2x - 3y = 12$
(h) $3x - 2y = 0$, $2x + 3y = 13$
(i) $2x + 3y = 19$, $5x + 2y = -2$
(j) $3x - 4y = 2$, $2x - 6y = -7$

5.3 Graphical Method of Solving Simultaneous Linear Equations

We have seen how we can use the two basic algebraic methods to solve simultaneous linear equations. What is the significance of the solution of a pair of simultaneous linear equations?

To find out, let us draw the **graphs** of the two equations in Example 7 on the same Cartesian plane. The pair of simultaneous linear equations in Example 7 are $2x + y = -1$ and $3x - 4y = 4$.

For the equation $2x + y = -1$, when $x = -2$,

$$\begin{aligned} 2x + y &= -1 \\ 2(-2) + y &= -1 \\ y &= = -1 + 4 \\ \therefore \quad y &= 3 \end{aligned}$$

$\Rightarrow (-2, 3)$

Continuing in this way for $x = 0$ and $x = 2$, we will get the points $(0, -1)$ and $(2, -5)$. Hence, the table of values for $\mathbf{2x + y = -1}$ is:

x	–2	0	2
y	3	–1	–5

note

Plotting any two points is sufficient to draw a straight line graph. However, we plot a third point as a check to make sure that there is no error.

Similarly, by substituting $x = -2$, 0 and 2 into the equation $\mathbf{3x - 4y = 4}$, we will get the following table of values:

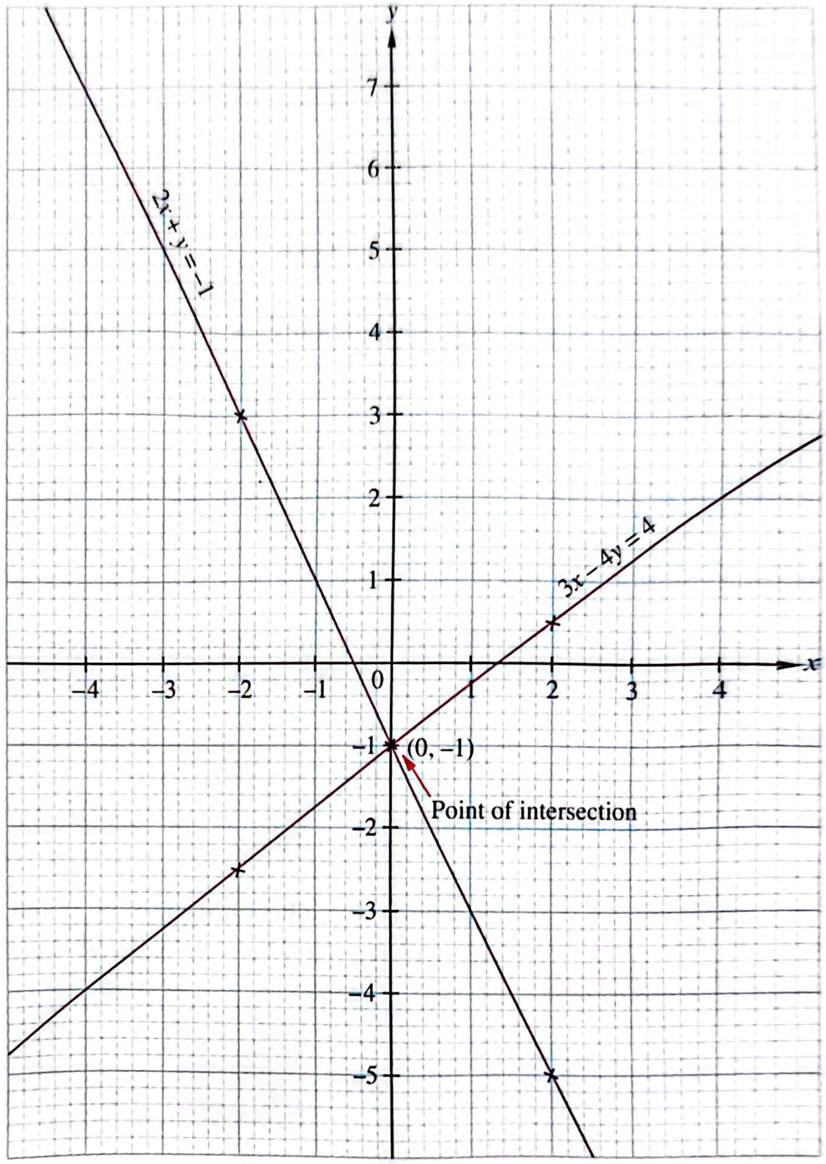
x	–2	0	2
y	–2.5	–1	0.5

Alternatively, for each of the above equations, we can express y in terms of x first and then use this equation to directly calculate the values of y. For example, for the equation $3x - 4y = 4$,

$$\begin{aligned} 3x - 4y &= 4 \\ 3x - 4 &= 4y \\ \frac{3x - 4}{4} &= y \\ \therefore y &= \frac{3}{4}x - 1 \end{aligned}$$

If the same values of x are chosen, the table of values for $y = \frac{3}{4}x - 1$ will be the same as that for $3x - 4y = 4$.

Next, plot the points in the table of values and draw the lines on a piece of graph paper as shown below.



Notice that the two lines intersect at the point (0, –1). The **solution** we obtained using the algebraic method in Example 7 was $x = 0$ and $y = -1$.

> The **point of intersection** of the two graphs representing the pair of simultaneous linear equations is its **solution**.

Example 9

Solve the following pair of simultaneous equations using the graphical method.

$$x - y = 2$$
$$2x + y = 10$$

Solution

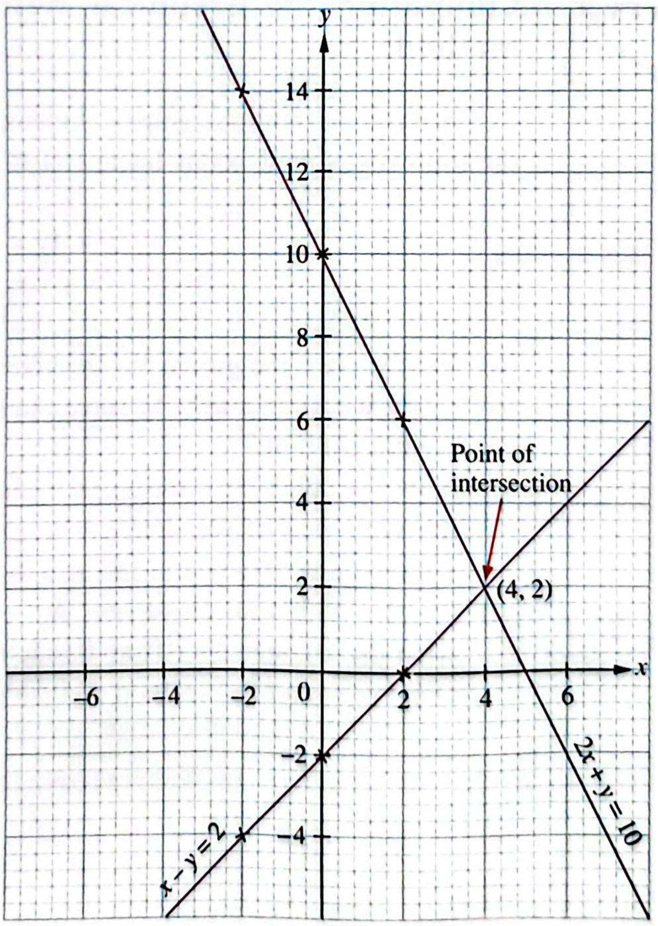
Table of values for $x - y = 2$:

x	−2	0	2
y	−4	−2	0

Table of values for $2x + y = 10$:

x	−2	0	2
y	14	10	6

Drawing the two graphs on a piece of graph paper, we get:



From the graph, the coordinates of the point of intersection are (4, 2). Therefore, the solution is $x = 4$ and $y = 2$.

note

Substitute the solution into both linear equations to check your answer, similar to what you have done in Section 5.2.

Exercise 5C

1. Solve the following pairs of simultaneous linear equations graphically.

   (a) $y = x - 2$
   $y = -2x + 4$

   (b) $y = 3x - 1$
   $y = 2x + 1$

   (c) $y = 3x$
   $y = x - 6$

   (d) $y = -2x$
   $y = -x + 5$

   (e) $y = \frac{5}{2}x$
   $y = 6x + 7$

   (f) $y = -\frac{1}{2}x + 3$
   $y = \frac{3}{2}x + 1$

2. Find the solutions to the following pairs of simultaneous linear equations by drawing their graphs.

   (a) $y = x - 4$
   $x + y = 6$

   (b) $y = x - 2$
   $3x + y = 6$

   (c) $x + y = 2$
   $2x - y = -5$

   (d) $x + y = 4$
   $4x - y = 1$

   (e) $x = 3$
   $x - y = 5$

   (f) $y = 5$
   $3x + y = -4$

   (g) $2x - y = 3$
   $3x + 2y = 1$

   (h) $x - 2y = -1$
   $x - y = -2$

TIME-OUT ACTIVITY

Comparing the Graphical Method and the Algebraic Method

Let us try to use both the graphical method and the algebraic method to solve the following simultaneous linear equations:

$$x + 4y = 14$$
$$2x + y = 4$$

(a) Firstly, draw the graphs for the above equations for $-2 \leq x \leq 2$.

(b) Then, use your graphs to solve the simultaneous linear equations. What is your solution?

(c) Now, use either the substitution method or the elimination method to solve the same pair of simultaneous linear equations. Compare your solution with the one obtained in **(b)**. Are the solutions the same?

(d) Discuss with your classmates and explain in your Mathematics Journal what you can conclude from this activity. Is there any difference between using the graphical method and the algebraic method?

5.4 No Solution and Infinitely Many Solutions

Activity

To study cases of simultaneous linear equations with no solution and infinitely many solutions.

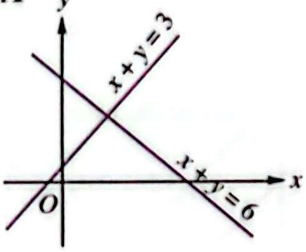
1. **(a)** Try solving the following simultaneous linear equations using either the substitution or the elimination method.

$$x + y = 3$$
$$x + y = 6$$

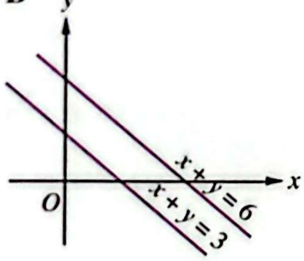
What do you notice?

(b) Next, draw the graphs of the two equations above on the same Cartesian plane. Which one of the three graphs below is the correct graph? Do the lines intersect each other?

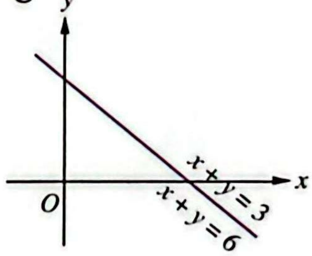
A



B



C



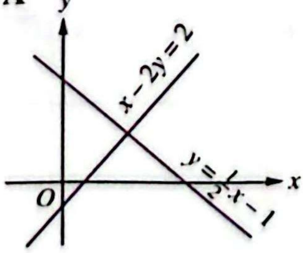
2. **(a)** Now, try solving this pair of simultaneous linear equations algebraically.

$$x - 2y = 2$$
$$y = \frac{1}{2}x - 1$$

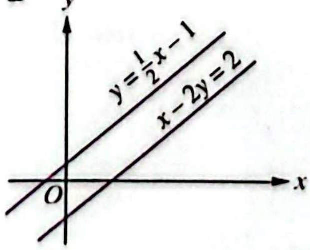
What do you notice?

(b) Next, draw the two lines on the same Cartesian plane. Which one of the three graphs below is the correct graph? How many points of intersections are there?

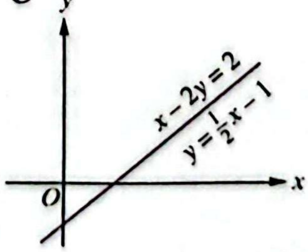
A



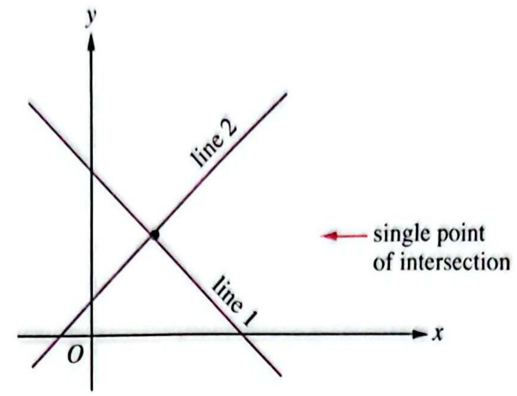
B



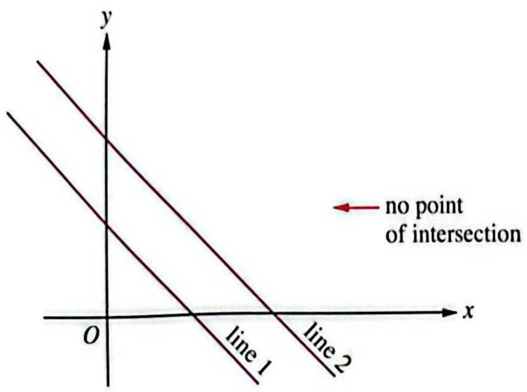
C



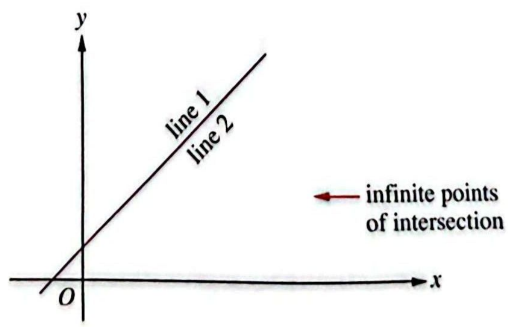
Most pairs of simultaneous linear equations have only one solution each. This can be shown by the single point of intersection when the graphs of the two equations are drawn.



In Question 1 of the activity on the previous page, we notice that the pair of simultaneous linear equations have **no solution**. This can be seen graphically as the two lines representing the equations are parallel and thus **do not intersect**.



In Question 2 of the activity, we can see an example of a pair of simultaneous linear equations that have **infinitely many solutions** (countless solutions). This can be seen graphically as the two lines representing the equations **overlap each other completely**.



5.5 Problem Solving Using Simultaneous Linear Equations

Many problems in everyday life involve more than one unknown quantity. Simultaneous equations can often be used to find the required values of these quantities. In this section, problems involving two unknowns will be discussed.

In such problems, firstly, the **unknown quantities** must be given appropriate letters to represent **variables**. The given information is then translated into **equations**. If there are two unknowns, there must be at least two pieces of information leading to two equations before the problem can be solved.

Example 10

The sum of two numbers is 20 and their difference is 2. Find the numbers.

Solution

Let the larger number be x and the smaller number be y.

The sum of the two numbers is 20.

i.e. $x + y = 20$ ....*(1)*

The difference of the numbers is 2.

i.e. $x - y = 2$ ....*(2)*

We write $x - y = 2$ and not $y - x = 2$ as $x > y$.

(1) + *(2)*,

$$(x + y) + (x - y) = 20 + 2$$
$$2x = 22$$
$$x = 11$$

note

The solutions must be translated back into the context of the original question and a final statement must be written to answer the question.

Substitute $x = 11$ into *(1)*,

$$11 + y = 20$$
$$y = 9$$

Therefore, the numbers are 11 and 9.

Example 11

The total cost of the tickets to a show for 2 adults and 3 children is \$16 while the cost for 3 adults and 2 children is \$19. Find the cost of an adult's ticket and that of a child's ticket.

Solution

Let the cost of an adult's ticket be $\$a$ and that of a child's ticket be $\$c$.

$$2a + 3c = 16 \quad \text{......} (1)$$
$$3a + 2c = 19 \quad \text{......} (2)$$

(1) × 2, $4a + 6c = 32$ ...... *(3)*
(2) × 3, $9a + 6c = 57$ ...... *(4)*

(4) – *(3)*, $5a = 25$
$a = 5$

Substitute $a = 5$ into *(1)*,

$$2(5) + 3c = 16$$
$$10 + 3c = 16$$
$$3c = 6$$
$$c = 2$$

Remember to check your answer. You can do this by substituting the solution into *(2)*.

Therefore, an adult's ticket costs \$5 and a child's ticket costs \$2.

Example 12

Five years ago, Mrs Wen was three times as old as her son, Shaohong. Five years from now, Mrs Wen's age will be twice her son's age. Find their present ages.

Solution

Let Mrs Wen's present age be x years and Shaohong's present age be y years.

Five years ago:
Mrs Wen's age was $(x - 5)$ years
and Shaohong's age was $(y - 5)$ years.

$$\therefore \quad x - 5 = 3(y - 5)$$
$$= 3y - 15$$
$$x = 3y - 10 \quad \text{......} (1)$$

Five years from now:
Mrs Wen's age will be $(x + 5)$ years
and Shaohong's age will be $(y + 5)$ years.

$$\therefore \quad x + 5 = 2(y + 5)$$
$$= 2y + 10$$
$$x = 2y + 5 \quad \text{.............................} \textit{(2)}$$

Substitute *(1)* into *(2)*,

$$3y - 10 = 2y + 5$$
$$3y - 2y = 5 + 10$$
$$y = 15$$

Substitute $y = 15$ into *(2)*,

$$x = 2(15) + 5$$
$$= 30 + 5$$
$$= 35$$

Therefore, Mrs Wen's present age is 35 years and Shaohong's present age is 15 years.

Exercise 5D

1. The sum of two numbers is 15 and their difference is 1. Find the numbers.

2. The sum of two numbers is 7. The difference between two times the larger number and the smaller one is 5. Find the two numbers.

3. Sam's Science mark is 15 less than his Maths mark. The total of his two marks is 145. Find Sam's mark for each subject.

4. The length of a rectangular label is 2 cm more than its width. If the perimeter of the label is 68 cm, find the length and the width.

5. The cost of 3 English books and 4 Maths books is \$78 while the cost of 2 English books and 3 Maths books is \$56. Find the cost of an English book and that of a Maths book.

6. In a farm, there are some cows and some chickens. If the animals have a total of 35 heads and 96 legs, how many cows and chickens are there respectively?

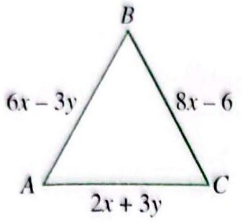
7. Four years ago, Ahmad was three times as old as Weihui. Four years from now, Ahmad will be only twice as old as Weihui. Find their present ages.

8. Five years ago, Ganesh was four times as old as his grandson, Ajit. Five years from now, Ganesh will be three times as old as Ajit. Find their present ages.

9. ABC is an equilateral triangle.

(a) Find the values of x and y.

(b) Hence, find its perimeter.



10. A shirt costs \$$s$ and a pair of trousers costs \$$t$.

It is given that $2s + t = 115$.

(a) Given also that three shirts and two pairs of trousers cost a total of \$200, write a second equation in s and t.

(b) Solve the two simultaneous equations to find the value of s and the value of t.

[N/92/P2]

TIME-OUT ACTIVITY

A Heavy Load

A pony and a donkey were carrying some bundles of cloth for a cloth merchant. On the way, the donkey said to the pony, "If I give you one bundle, my load will be half of yours." In an angry mood, the pony retorted, "Why don't you take one bundle from me, then my load will be half of yours!"

Assuming that the bundles are of equal weight, how many bundles did the pony and the donkey each carry?

SUMMARY

1. The pair of numbers which satisfy a pair of simultaneous linear equations is called the solution of the equations.

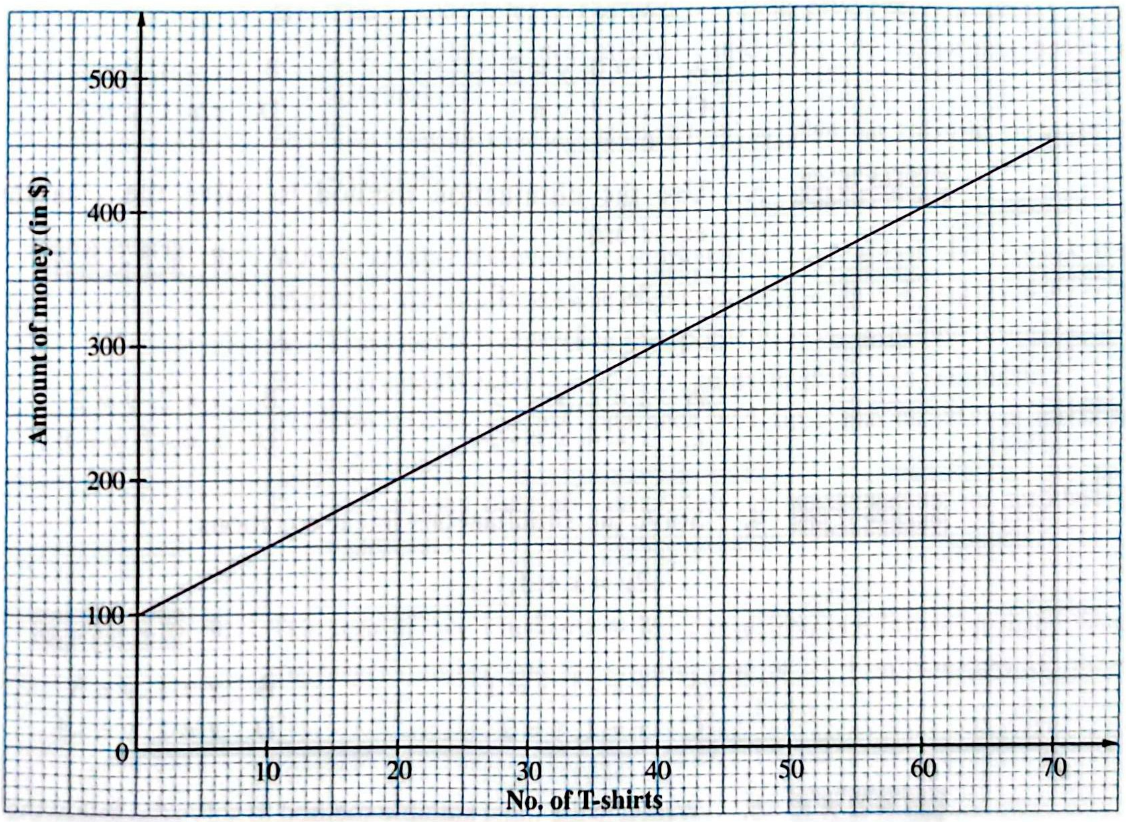
2. The two algebraic methods commonly used in solving simultaneous linear equations are
   (a) the substitution method,
   (b) the elimination method.

3. The graphical method can also be used to solve simultaneous linear equations. The point of intersection of the two lines representing the pair of equations is its solution.

A Business Graph

Chong Ming plans to set up a stall selling T-shirts during a special fair at the community centre near his home. The stall rental is fixed at \$100 per day. Each T-shirt costs him \$5 and he intends to sell each for \$8. It is assumed that Chong Ming is able to sell off all the T-shirts he had bought.

The graph below shows the relationship between Chong Ming's total expenses and the number of T-shirts he buys.



(a) By letting the total expenses be Y and the number of T-shirts bought by Chong Ming be X, show that the equation represented by the above graph is

$$Y = 5X + 100$$

(b) By letting the revenue from the sales of T-shirts be y and the number of T-shirts sold by Chong Ming be x, form another equation that relates the revenue and the number of T-shirts sold.

(c) Copy the above graph on a graph paper. Then, draw the line that represents the equation in **(b)** on the same axes.

(d) Deduce the minimum number of T-shirts that he must sell in order to break even.

(e) Deduce the number of T-shirts he must sell in order to earn a profit of \$140. (Profit = revenue – expenses)

10 MINUTES
CONCEPT CHECK

You have 10 minutes to answer the following questions.
Choose the most appropriate answer.

5.1 **1.** Which one of the following satisfies both linear equations $2x + y = 5$ and $x - y = -2$?

A $x = 0, y = 5$ **B** $x = 2, y = 4$
C $x = 1, y = 3$ **D** $x = 5, y = 3$

5.2 **2.** Use the substitution method to solve the simultaneous linear equations

$$x + 2y = 4,$$
$$x = 2y.$$

A $x = 0, y = 2$ **B** $x = 4, y = 2$
C $x = 2, y = 1$ **D** $x = 1, y = 2$

5.2 **3.** Use the elimination method to solve the simultaneous linear equations

$$3x + 2y = 10,$$
$$5x - 2y = 6.$$

A $x = 2, y = -2$ **B** $x = 2, y = 2$
C $x = 4, y = -2$ **D** $x = -2, y = 8$

5.2 **4.** Solve the simultaneous linear equations

$$2y = x + 3,$$
$$y = x + 2.$$

A $x = -1, y = 1$ **B** $x = 1, y = 3$
C $x = 1, y = 2$ **D** $x = 1, y = -1$

5.2/5.3 **5.** Which is the most suitable method to solve the following simultaneous linear equations?

$$x = 5 - 2y$$
$$3x - y = -1$$

A Substitution method **B** Elimination method
C Graphical method **D** Trial and error

5.3/5.4 **6.** Which of the following statements is true about the graphs that represent the simultaneous linear equations $x + y = 4$ and $x - y = 2$?

A The graphs intersect at one point to give only one solution.
B The graphs intersect at two points to give two solutions.
C The graphs do not intersect and thus the equations have no solution.
D The graphs overlap completely to give an infinite number of solutions.

5.3/5.4 **7.** Which of the following statements is true about the graphs that represent the simultaneous linear equations $x + y = 1$ and $x + y = 5$?

A The graphs intersect at one point to give only one solution.
B The graphs intersect at two points to give two solutions.
C The graphs do not intersect and thus the equations have no solution.
D The graphs overlap completely to give an infinite number of solutions.

5.3/5.4 **8.** Which of the following statements is true about the graphs that represent simultaneous linear equations $x + 2y = 4$ and $2x + 4y = 8$?

A The graphs intersect at one point to give only one solution.
B The graphs intersect at two points to give two solutions.
C The graphs do not intersect and thus the equations have no solution.
D The graphs overlap completely to give an infinite number of solutions.

5.5 **9.** The sum of two numbers is 16. The difference between three times the larger number and the smaller one is 20. Find the two numbers.

A −1, 17 **B** 2, 14
C 7, 9 **D** 18, −2

5.5 **10.** The cost of 3 pens and 4 pencils is \$5 while the cost of 2 pens and 2 pencils is \$3. What is the cost of a pen?

A \$0.50 **B** \$1.00
C \$1.50 **D** \$2.00

Revision Paper 5

SECTION A

1. Solve the simultaneous linear equations

$$5y = 3x + 4,$$
$$5y = 4x - 3.$$

2. Solve the simultaneous linear equations

$$3x + 4y = 5,$$
$$y = x - 4.$$

3. Solve the simultaneous linear equations

$$x + y = 4,$$
$$x - y = -2.$$

4. Solve the simultaneous linear equations

$$3x + y = 22,$$
$$2x + y = 17.$$

SECTION B

5. Solve the simultaneous linear equations

$$5x + 2y = 16,$$
$$7x - 3y = 5.$$

6. The sum of two numbers is 30 and their difference is 4. Form a pair of simultaneous linear equations and find the numbers.

7. 6 kg of mutton and 1 chicken cost \$60 while 8 kg of mutton and 5 chickens cost \$124. Find the cost of 1 kg of mutton and 1 chicken respectively.

8. **(a)** Copy and complete the table of values for $y = 4x - 3$ and $y = -2x + 3$.

(i) Table of values for $y = 4x - 3$:

x	0	2	4
y	−3		

(ii) Table of values for $y = -2x + 3$:

x	0	2	4
y		−1	

(b) Using a scale of 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis, draw the graphs of $y = 4x - 3$ and $y = -2x + 3$ for $0 \leq x \leq 4$.

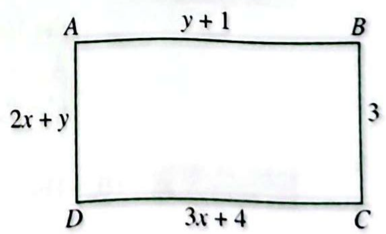
(c) Use your graphs to solve the simultaneous linear equations

$$y = 4x - 3,$$
$$y = -2x + 3.$$

SECTION C

9. Two years ago, Liza was three times as old as Petrina. Three years from now, Liza will only be twice as old as Petrina. Form a pair of simultaneous equations and find their present ages.

10. *ABCD* is a rectangle.



(a) Find the values of x and y.
(b) Hence, find the area of the rectangle.

Chapter

6

Triangles, Polygons and Congruence

DISCOVER!

How to:

- construct simple geometrical figures
- classify triangles and special quadrilaterals according to their properties
- apply properties of triangles and special quadrilaterals to calculate unknown angles
- construct various types of triangles and quadrilaterals
- find the angle sum of the interior and exterior angles of any convex polygon
- recognise congruent figures
- match sides and angles of two congruent polygons

Take a look at the design of buildings around you. Are they always rectangular or round? Can you find buildings that take the form of other shapes?

The tall building shown in the picture above is in the form of an octagon which is an eight-sided plane figure. Octagons belong to a group of geometric figures called polygons. Polygons, by definition, are plane figures that have three or more line segments as their sides. They are named according to the number of sides that they have; triangles being the simplest type of polygons. Eight-sided polygons are thus named 'octagons' as the prefix 'octa' stands for eight. Octagons are especially favoured by the Chinese in the design of buildings. Do you know why?

In this chapter, you will learn more about different types of polygons and their properties.

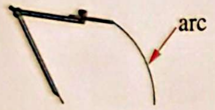
6.1 Simple Geometrical Constructions

6.1.1 Angle Bisectors

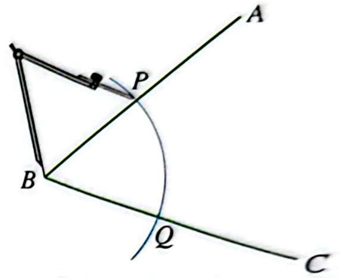
In Book 1, we have seen how an **angle** is formed when two line segments or rays meet at a point. An additional line segment or ray can be drawn to **bisect an angle** (i.e., divide an angle into two equal parts). Let us explore how we can bisect any angle using a pair of compasses and a ruler.

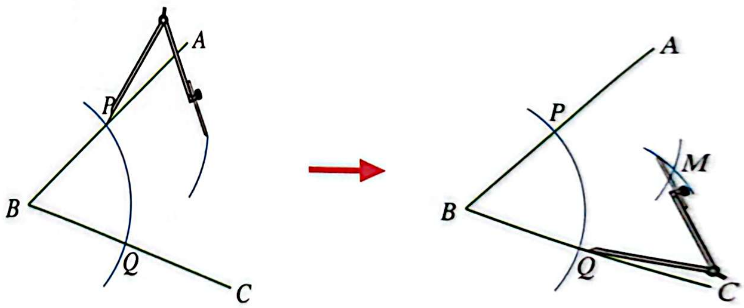
Activity

To bisect an angle using a pair of compasses and a ruler and to explore its properties.

> **note**
>
> An **arc** is a part of the circumference of a circle.
>
> 
>

Step	*Action*
1	Construct an acute angle and label it as $\angle ABC$.
2	With the compass point at vertex B and a suitable radius, draw an arc to cut the two rays of the angle. Label these points as P and Q.
3	Next, draw two more arcs with the same radius, using P and Q as the centres as shown in the diagrams below. Label the point where the two arcs intersect as M.





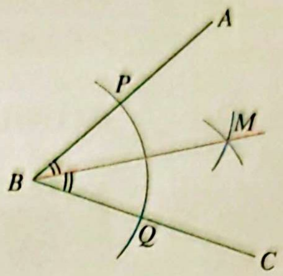
> **note**
>
> Use a sharp pencil to do any construction and to draw lines. All construction lines and arcs must be clearly and neatly shown and must not be erased away.

Step	*Action*
4	Join BM.
5	Use a protractor to measure $\angle ABM$ and $\angle CBM$. Are the two angles equal?
6	Repeat steps 1–5 with an obtuse angle.

An applet that shows the construction of an angle bisector can be found at the following website: http://illuminations.nctm.org/ActivityDetail.aspx?ID=49

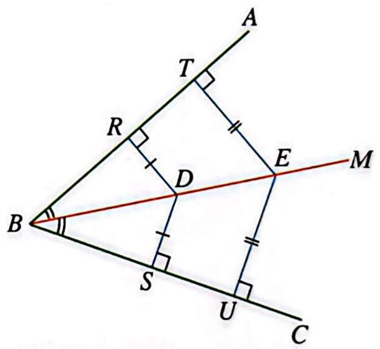
From the activity on the previous page, we can see the following:

$\angle ABM = \angle CBM = \frac{1}{2}\angle ABC.$



We can thus say that the line BM bisects $\angle ABC$. We call this line the **angle bisector** of $\angle ABC$.

Notice also, from the diagram shown below, that if we mark a point D on the angle bisector and draw a perpendicular line from point D to each of the arms of the angle and label them R and S respectively, then $DR = DS$.



The same can be observed if we take another point E on the angle bisector. In this case, $ET = EU$, as can be seen on the diagram shown above. We say that the points D and E are equidistant from the arms of the angle.

Try this with the diagram that you have drawn in the previous activity. Does this hold true for any point on the angle bisector?

In general,

1. The **angle bisector** of an angle is the line segment or ray that bisects the angle.
2. We can construct an angle bisector using a pair of compasses and a ruler.
3. Any point on the angle bisector of an angle is **equidistant** from the arms of the angle.

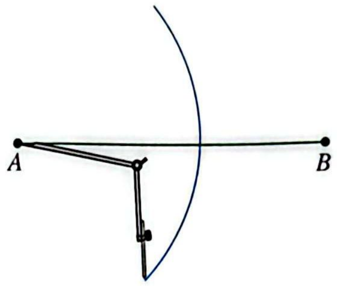
6.1.2 Perpendicular Bisectors

We have just learned how to bisect an angle using a pair of compasses and a ruler. We can also use a pair of compasses and a ruler to bisect a line segment.

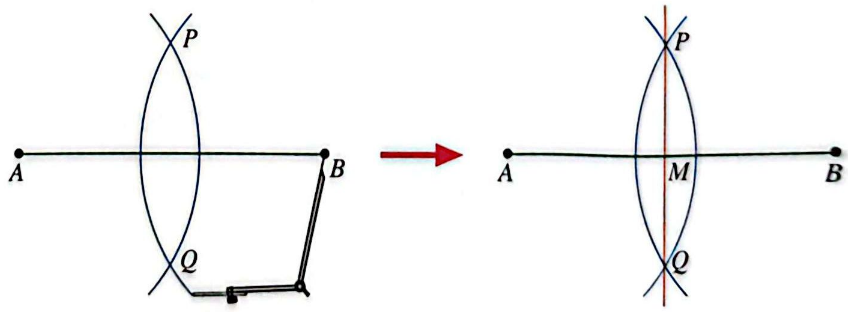
Activity

To bisect a line segment using a pair of compasses and a ruler and to explore its properties.

Step	*Action*
1	Draw a line and label its end-points *A* and *B* respectively. (The line segment that you draw should preferably be at least 4 cm long.)
2	With *A* as the centre and a radius that is slightly more than half of the length of the line *AB*, draw an arc as shown in the diagram below.



3	With *B* as the centre and the same radius as in step 2, draw an arc to intersect the first arc at two points. Label these points as *P* and *Q*.
4	Join *PQ* and label the point where *PQ* cuts *AB* as *M*.

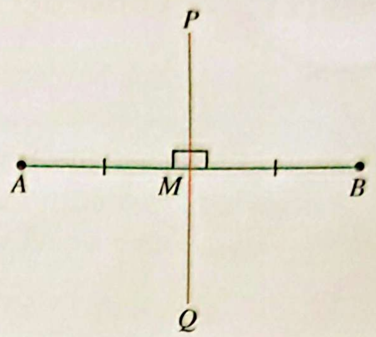


5	Measure *AM* and *MB*. Are the line segments *AM* and *MB* of the same length?
6	Use a protractor to measure $\angle AMP$ and $\angle BMP$. What can you say about $\angle AMP$ and $\angle BMP$?

From the activity on the previous page, we can see the following:

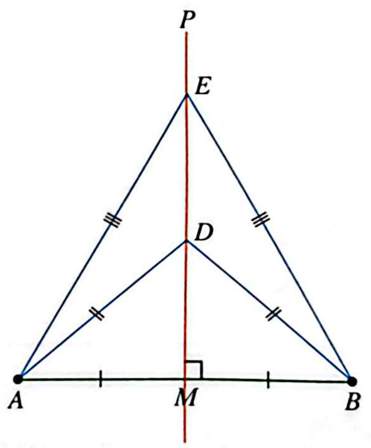
$AM = MB = \frac{1}{2}AB$

$\angle AMP = \angle BMP = 90°$



We can thus say that the line PQ bisects the line AB at right angles. We call this line the **perpendicular bisector** of the line AB.

Notice also, from the diagram shown below, that if we mark a point D on the perpendicular bisector and draw a line from it to each of the end-points of the line segment AB, then $AD = BD$.



The same can be observed if we take another point E on the perpendicular bisector. In this case, $AE = BE$ as can be seen on the diagram shown above. We say that the points D and E are equidistant from the two end-points of the line segment.

Try this with the diagram that you have drawn in the previous activity. Does this hold true for any point on the perpendicular bisector?

In general,

1. The **perpendicular bisector** of a line segment is the line that bisects the line segment and is perpendicular to it.
2. We can construct a perpendicular bisector using a pair of compasses and a ruler.
3. Any point on the perpendicular bisector of a line segment is **equidistant** from the two end-points of the line segment.

TIME-OUT ACTIVITY

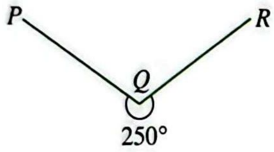
Construct Us!

You have learned how an angle bisector and a perpendicular bisector can each be constructed using a pair of compasses and a ruler. Try constructing each of them in another way. (For example, try using a protractor and a ruler instead to construct them. You can also try using the GSP software to do this.) Explain the method you use and the steps in detail in your Mathematics Journal.

Exercise 6A

1. **(a)** Use a protractor to draw angles of following sizes.
(i) 45° **(ii)** 120°
(b) Bisect each angle using only a pair of compasses and a ruler. Check the angles in the two halves with a protractor.

2. **(a)** Draw the reflex angle PQR given below.



(b) Construct its angle bisector using a pair of compasses and a ruler. Check the angles in the two halves with a protractor.

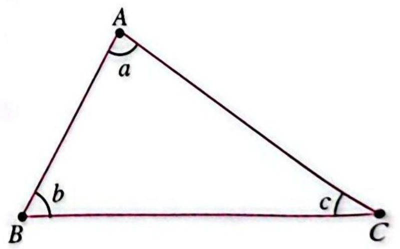
3. **(a)** Draw lines of the following lengths.
(i) 8 cm **(ii)** 10.4 cm
(b) Bisect each line using only a pair of compasses and ruler. Check the lengths of the two halves.

6.2 Triangles and Their Properties

6.2.1 Types of Triangles

As you have learned before, a **triangle** is a closed plane figure with three sides.

Look at the triangle shown on the right. The points A, B and C are called the **vertices** (singular: **vertex**) of the triangle.



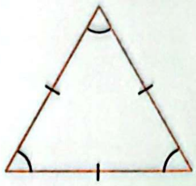
We can refer to the given triangle as triangle ABC or $\triangle ABC$ where '$\triangle$' stands for 'triangle'.

The angles, a, b and c, which lie inside the triangle ABC are known as its **interior angles**.

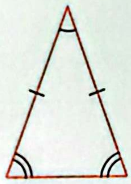
Triangles can be classified according to the lengths of their sides or the sizes of their angles.

Classifying by sides:

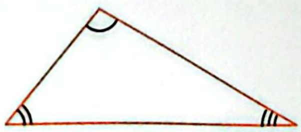
1. A triangle with all sides of equal length is known as an **equilateral triangle**. All its angles are equal.



2. A triangle with two sides of equal length is known as an **isosceles triangle**. Its base angles, which are the two angles opposite the equal sides, are equal.

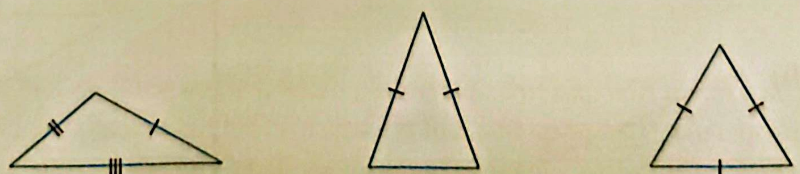


3. A triangle with all sides of different lengths is known as a **scalene triangle**. All its angles are different.



We have seen that we can identify the different types of triangles by measuring the lengths of their sides. In some triangles, however, the sides are already marked with the following symbols: -|-, -||- or -|||-. Sides which are of equal length are marked with the same symbol. Thus, in such cases, we do not need to measure the lengths of the sides in order to identify the type of triangle.

E.g.

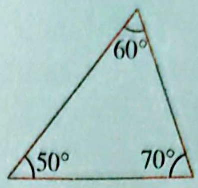


scalene triangle | **isosceles triangle** | **equilateral triangle**

Classifying by angles:

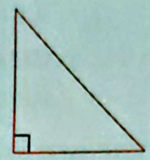
1. A triangle with all angles measuring less than 90° is called an **acute-angled triangle**.

   E.g.



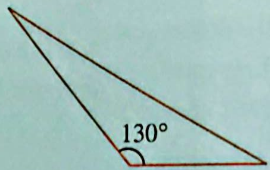
2. A triangle with one angle measuring exactly 90° is called a **right-angled triangle**.

   E.g.



3. A triangle with one angle measuring greater than 90° is called an **obtuse-angled triangle**.

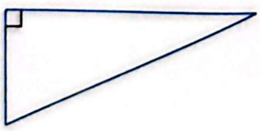
   E.g.



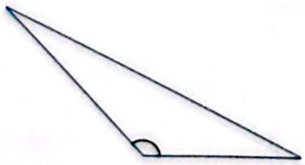
Exercise 6B

1. Name each of the following triangles according to
   (i) the lengths of its sides,
   (ii) the sizes of its angles.

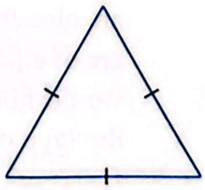
(a)



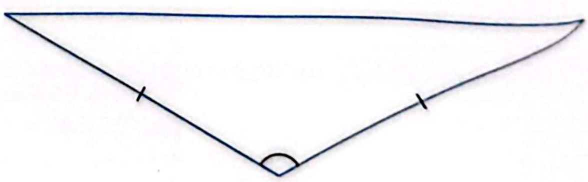
(b)



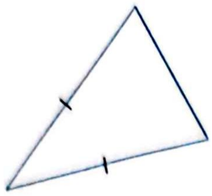
(c)



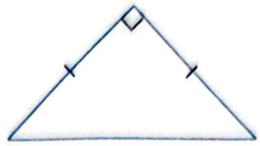
(d)



(e)



(f)

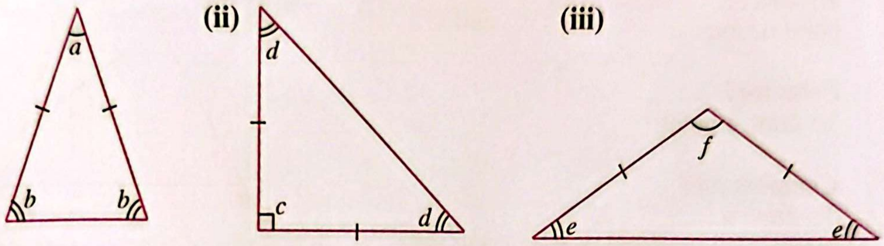


TIME-OUT ACTIVITY

Name Us!

Study the three triangles below.

(i) (ii) (iii)

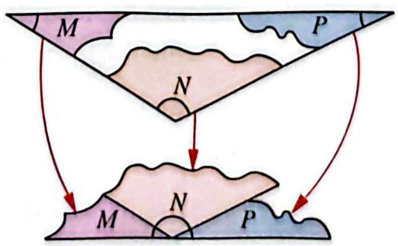


(a) Give an appropriate name to each of the triangles above such that each triangle is classified both by its side properties and angle properties. Can you pick out the common property that the three triangles have?

(b) Is it possible to have equilateral triangles named in three different ways as you have seen in **(a)**? Explain your reasons.

6.2.2 Sum of the Angles of a Triangle

Draw a triangle and label each angle with a letter. Next, tear off each corner and place them side by side with their vertices together.



Notice that all three angles, when placed side by side, will lie on a straight line. Since adjacent angles on a straight line add up to 180°, we can deduce that the sum of the angles in the triangle is 180°.

Let us use the Geometer's Sketchpad (GSP) to investigate whether the above property applies to all triangles.

Activity

To study the property relating to the sum of the angles in a triangle using Geometer's Sketchpad (GSP).

The GSP is a powerful IT tool for geometrical construction.

A quick guide of the basic tools in GSP is given below.

Selection Arrow tool
To select a point or line.

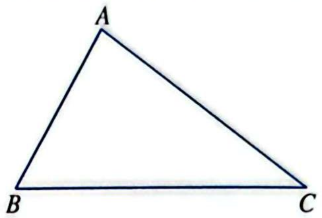
Point tool
To draw a point.

Compass tool
To draw a circle.

Straightedge tool
To draw a straight line.

Text tool
To label a point, line, etc.

Step	***Action***
1	To access the program, double-click on the GSP icon on the desktop.
2	Click on the *Straightedge tool* and drag right to choose the *Line Segment tool*.
3	Draw three line segments in succession to form a triangle.
4	Click on the *Text tool* to label the vertices of the triangle as *A*, *B* and *C*.



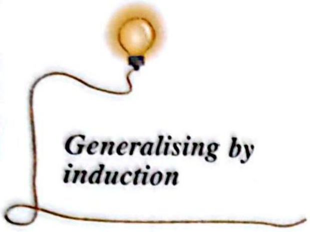
5	Click on the *Selection Arrow tool*. Holding the *Shift* key down, click on the points *C*, *A* and *B* in succession before releasing the *Shift* key.
6	Click on *Measure* from the *Menu Bar* and choose *Angle*. (Note: The angle *CAB* will be displayed on the screen as '$m\angle CAB =$ ')
7	Repeat steps 5 and 6 for the points *A*, *B* and *C*. (Note: The angle *ABC* will be displayed on the screen as '$m\angle ABC =$ ')
8	Repeat steps 5 and 6 for the points *B*, *C* and *A*. (Note: The angle *BCA* will be displayed on the screen as '$m\angle BCA =$ ')
9	Click on *Measure* from the *Menu Bar* and choose *Calculate*. (A calculator will be shown on the screen.)
10	Click on: '$m\angle CAB =$ ' (on the main screen), '+' (on the calculator screen), '$m\angle ABC =$ ' (on the main screen), '+' (on the calculator screen), '$m\angle BCA =$ ' (on the calculator screen) and then *OK* (on the calculator screen).

What can you observe about the value of '$m\angle CAB + m\angle ABC + m\angle BCA$'?

11	Use the *Selection Arrow tool* to change the sizes of the angles by dragging any one of the vertices of the triangle.
12	Click on *File* and choose the *Save As* command from the *Menu Bar*. Enter the filename as 'TriSum.gsp' in the *Save As* window and then click on *Save*.

Does your earlier observation hold true even when the values of the angles are changed?

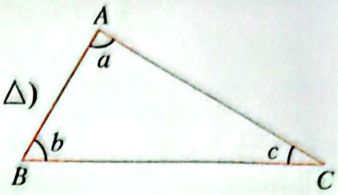
By induction, the results of the activity on the previous page can be generalised as follows:



Generalising by induction

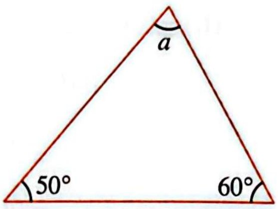
The **angle sum of a triangle** is 180°,

i.e. $\angle a + \angle b + \angle c = 180°$ **($\angle$ sum of $\triangle$)**



Example 1

Find the value of $\angle a$ in the figure shown below.

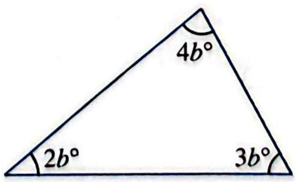


Solution

$$\angle a + 50° + 60° = 180° \quad (\angle \text{ sum of } \triangle)$$
$$\therefore \quad \angle a = 180° - (50° + 60°)$$
$$= 180° - 110°$$
$$= 70°$$

Example 2

Find the value of b in the figure shown below.



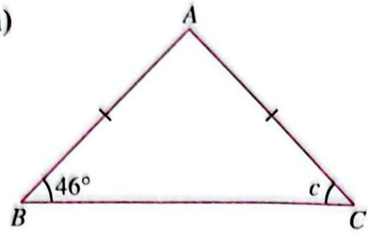
Solution

$$2b° + 3b° + 4b° = 180° \quad (\angle \text{ sum of } \triangle)$$
$$9b° = 180°$$
$$\frac{9b°}{9} = \frac{180°}{9}$$
$$b° = 20°$$
$$\Rightarrow b = 20$$

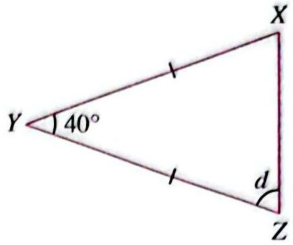
Example 3

Find the value of the unknown angle marked in each of the triangles given below.

(a)



(b)



Solution

(a) $\triangle ABC$ is an isosceles triangle,

$\therefore \quad \angle c = 46°$ (base $\angle$s of isos. $\triangle ABC$)

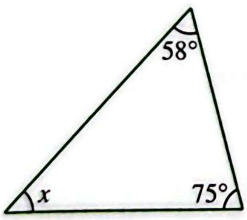
(b) $\triangle XYZ$ is an isosceles triangle,

$$\therefore \quad \angle d = \frac{180° - 40°}{2} \quad \text{(base } \angle\text{s of isos. } \triangle XYZ\text{)}$$
$$= \frac{140°}{2}$$
$$= 70°$$

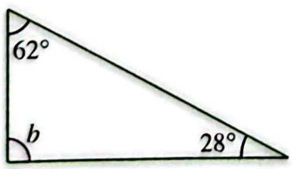
Exercise 6C

1. Find the value of the unknown angle(s) marked in each of the following figures. The figures shown are not drawn to scale.

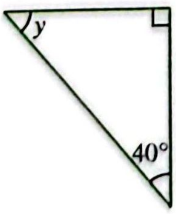
(a)



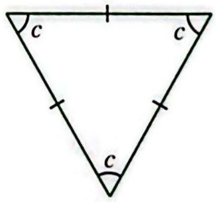
(b)



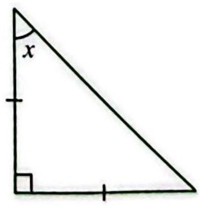
(c)



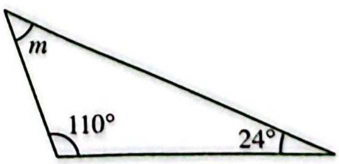
(d)



(e)

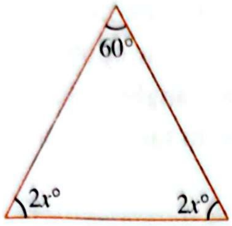


(f)

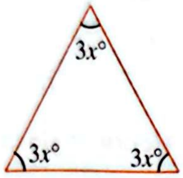


2. Find the value of x in each of the following figures. The figures shown are not drawn to scale.

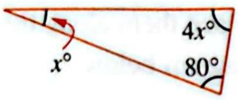
(a)



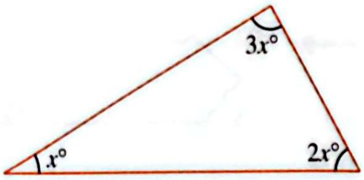
(b)



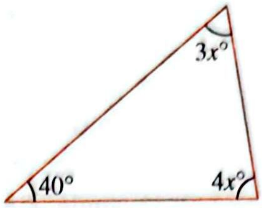
(c)



(d)



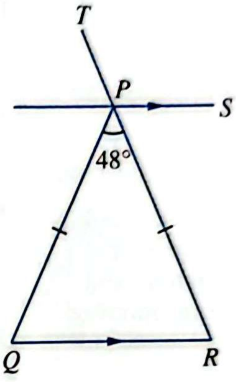
(e)



3. PQR is an isosceles triangle in whic $PQ = PR$ and $\angle QPR = 48°$. PS is parallel QR and TPR is a straight line. Calculate

(a) $\angle QRP$,

(b) $\angle TPS$.

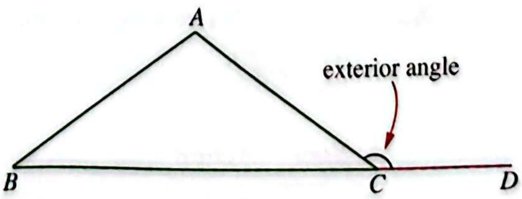


6.2.3 Exterior Angle of a Triangle

When a side of a triangle is **produced** (extended), an angle is formec between the produced side and one of the sides of the triangle. We cal such an angle an **exterior angle**. In the diagram given below, $\angle ACD$ is an exterior angle of $\triangle ABC$.

note

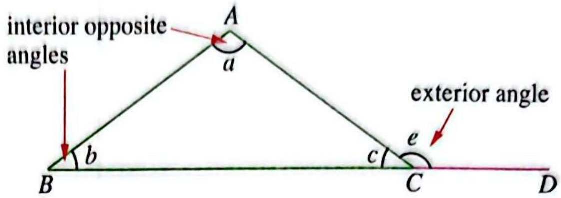
Interior angles of a triangle lie **inside** the triangle.
Exterior angles of a triangle lie **outside** the triangle.



Besides $\angle ACD$, triangle ABC can have several other exterior angles. Can you draw and name some of them?

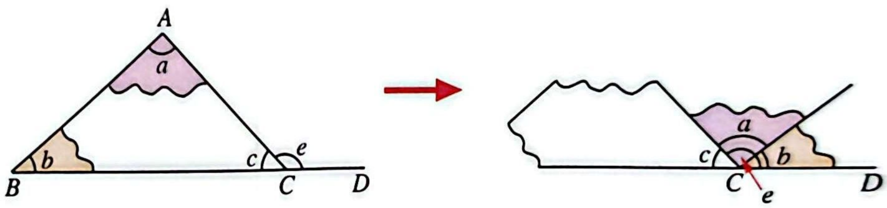
Recall that angles that lie within a triangle are known as its **interior angles**.

In the diagram shown below, the interior angles are marked a, b and c. Notice that two of the interior angles, $\angle a$ and $\angle b$, lie opposite the exterior angle, $\angle e$. They are called the **interior opposite angles** with respect to $\angle e$. The remaining interior angle, $\angle c$, is adjacent to the exterior angle, $\angle e$.



Let us now explore the relationship between an exterior angle of a triangle and its two interior opposite angles.

Draw a triangle and label each angle with a letter. Next, produce one of it sides. Tear off the corners at A and B and place them along the produced side next to $\angle c$, as shown in the second diagram below.



You will see that the two angles, $\angle a$ and $\angle b$, add up to give the exterior angle, $\angle e$. We can thus deduce that an exterior angle of a triangle is equal to the sum of its two interior opposite angles.

We shall next use the Geometer's Sketchpad (GSP) to investigate whether the above property applies to all triangles.

note

Another way of deducing the relationship between the exterior angle of a triangle and its two interior opposite angles:

$\angle a + \angle b + \angle c$
$= 180°$ ($\angle$ sum of Δ)

and

$\angle e + \angle c$
$= 180°$ (adj. $\angle$s on a str. line)

$\therefore \angle e = \angle a + \angle b$
(ext. $\angle$ of Δ)

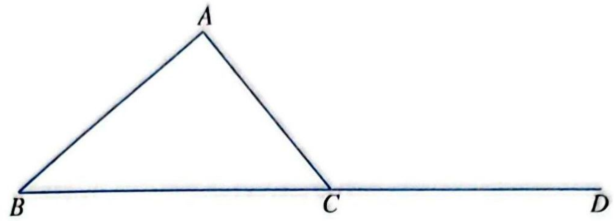
Activity

To study the relationship between an exterior angle of a triangle and its two interior opposite angles using Geometer's Sketchpad (GSP).

Step	*Action*
1	Open the GSP file [TriSum.gsp] that you have saved from the previous activity on page 182.
2	Click on the *Selection Arrow tool*. Holding the *Shift* key down, click on the points B and C in succession before releasing the *Shift* key.
3	Click on *Transform* from the *Menu Bar* and choose *Mark Vector 'B->C'*.

4 Holding the *Shift* key down, click on the line segment *BC* before releasing the *Shift* key.

5 Click on *Transform* from the *Menu Bar* and choose *Translate*. Then, click *OK* and relabel the point *C'* as *D*.



6 Holding the *Shift* key down, click at the points *A*, *C* and *D* in succession before releasing the *Shift* key.

7 Click on *Measure* from the *Menu Bar* and choose *Angle*. (Note: The angle *ACD* will be displayed on the screen as '$m\angle ACD =$ ')

8 Repeat steps 6 and 7 for *A*, *B* and *C*. (Note: The angle *ABC* will be displayed on the screen as '$m\angle ABC =$ ')

9 Repeat steps 6 and 7 for *C*, *A* and *B*. (Note: The angle *CAB* will be displayed on the screen as '$m\angle CAB =$ ')

10 Click on *Measure* from the *Menu Bar* and choose *Calculate*. (A calculator is shown on the screen.)

11 Click on:
'$m\angle ABC =$ ' (on the main screen), '+' (on the calculator screen), '$m\angle CAB =$ ' (on the main screen) and then *OK* (on the calculator screen).

What can you observe about the value of '$m\angle ABC + m\angle CAB =$ ' and the value of '$m\angle ACD =$ '?

12 Use the *Selection Arrow tool* to change the sizes of the angles by dragging any one of the vertices of the triangle.

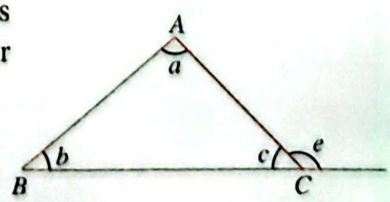
Does your earlier observation hold true even when the values of the angles are changed?

Generalising by induction

By induction, the results of the above activity can be generalised as follows:

An **exterior angle of a triangle** is equal to the sum of the two interior opposite angles,

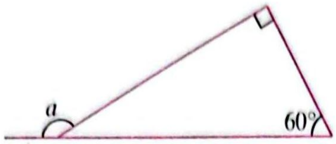
i.e. $\angle e = \angle a + \angle b$ **(ext. $\angle$ of Δ)**



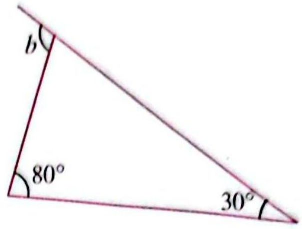
Example 4

Find the value of the unknown angle marked in each of the figures given below.

(a)



(b)



Solution

(a) $\angle a = 90° + 60°$ (ext. $\angle$ of Δ)

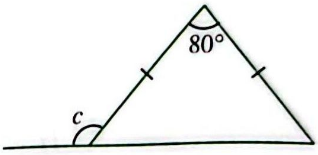
$= 150°$

(b) $\angle b = 80° + 30°$ (ext. $\angle$ of Δ)

$= 110°$

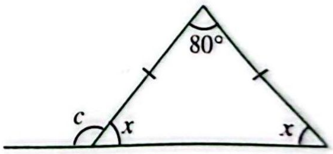
Example 5

Find the value of $\angle c$ in the figure shown on the right.



Solution

Let the base angles of the triangle be x.



$\angle x = \dfrac{180° - 80°}{2}$ (base $\angle$s of isos. Δ)

$= \dfrac{100°}{2}$

$= 50°$

$\therefore \quad \angle c = 80° + 50°$ (ext. $\angle$ of Δ)

$= 130°$

note

Since $\angle c$ and $\angle x$ lie on a straight line. We can also find $\angle c$ in the following way:

$\angle c + \angle x = 180°$ (adj. $\angle$s on a str. line)

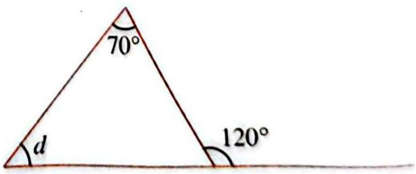
$\therefore \ \angle c = 180° - \angle x$

$= 180° - 50°$

$= 130°$

Example 6

Find the value of $\angle d$ in the figure shown on the right.



Solution

$120° = 70° + \angle d$ (ext. $\angle$ of Δ)

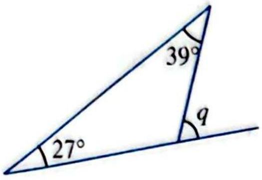
$\therefore \quad \angle d = 120° - 70°$

$= 50°$

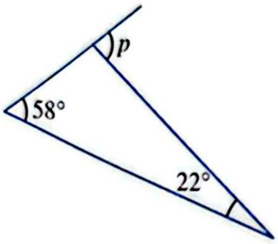
Exercise 6D

1. Find the value of the unknown angle marked in each of the following figures.

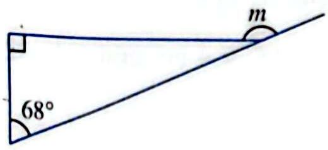
(a)



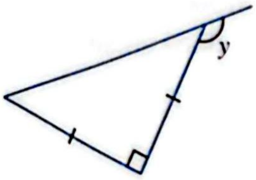
(b)



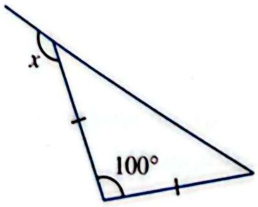
(c)



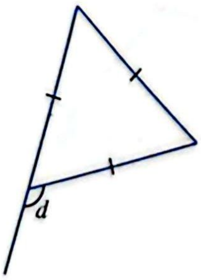
(d)



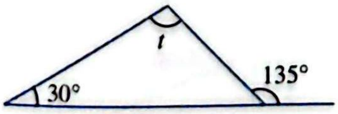
(e)



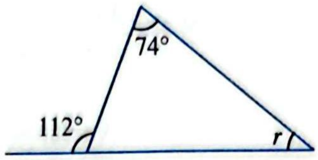
(f)



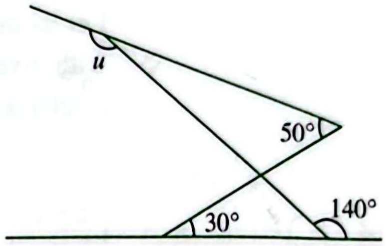
(g)



(h)

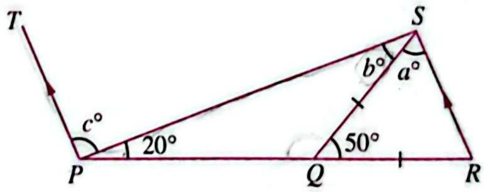


2. Find the size of the angle labelled u in the following figure.



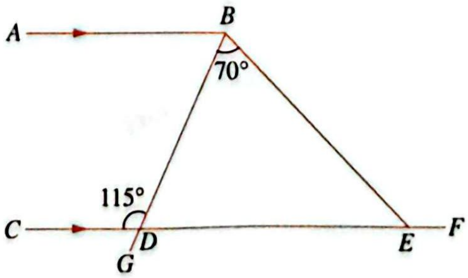
3. In the diagram given below, PQR is a straight line, $QR = QS$ and TP is parallel to SR. $\angle RQS = 50°$ and $\angle QPS = 20°$. Find the values of

(a) a, **(b)** b, **(c)** c.



4. The lines AB and $CDEF$ are parallel. Angle CDB = 115° and angle DBE = 70°. Calculate

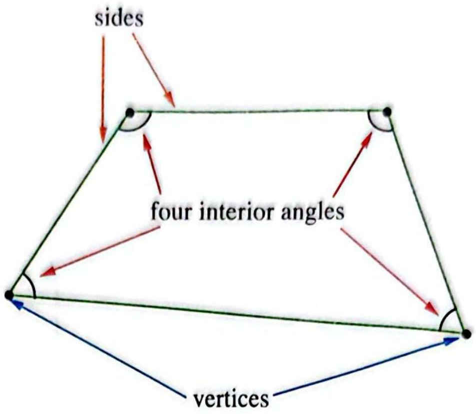
(a) angle GDE,
(b) angle ABD,
(c) angle BEF.



[N/95/P1]

6.4 Quadrilaterals and Their Properties

A **quadrilateral** is a closed plane figure with four sides joined by four **vertices**. Thus, a quadrilateral has four interior angles.

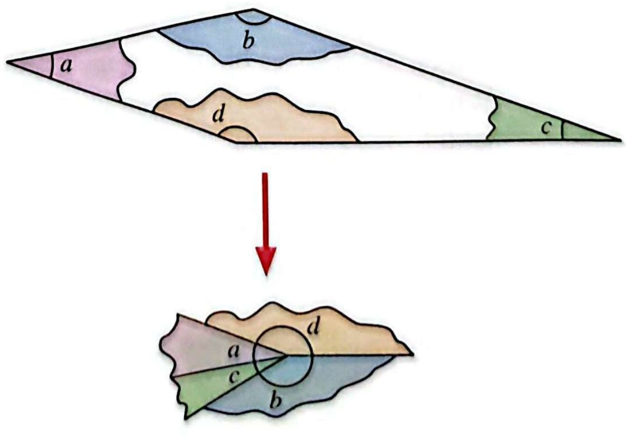


Some examples of quadrilaterals, which you may have already come across before, are the parallelogram, rectangle, rhombus, square and trapezium.

6.4.1 Sum of the Angles of a Quadrilateral

We can find the sum of the angles of a quadrilateral in the same way as we did for a triangle in Section 6.2.2.

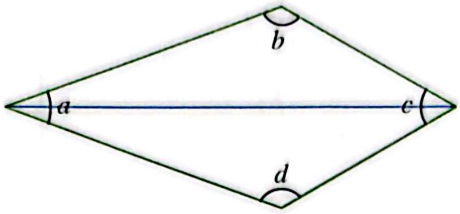
By labelling each angle of the quadrilateral with a letter, tearing off each corner and then placing their vertices together, we can observe that the four angles form a full turn, i.e., 360°.



Therefore, the sum of the angles of a quadrilateral is 360°.

Alternatively, if we draw a line from one vertex to the opposite vertex in the quadrilateral, we will obtain two triangles as shown below.

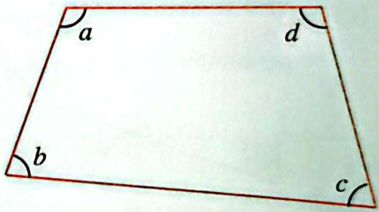
A line segment that joins two opposite vertices of the quadrilateral together is called a **diagonal**. Since a quadrilateral has two pairs of opposite vertices, there are two diagonals in a quadrilateral. You will learn more about the properties of diagonals in special quadrilaterals in the next section.



Since we know that the sum of the angles of a triangle is 180°, we can deduce that the sum of the angles of a quadrilateral is $2 \times 180° = 360°$.

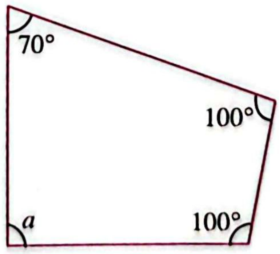
The **sum of the angles of a quadrilateral** is 360°,

i.e. $\angle a + \angle b + \angle c + \angle d = 360°$ (**∠ sum of quad.**)



Example 11

Find the unknown angle marked in the quadrilateral given below.



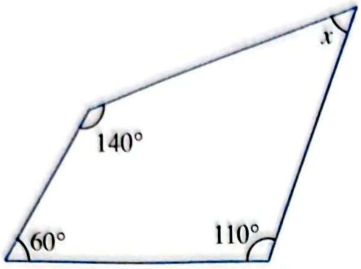
Solution

$$\begin{aligned}\angle a &= 360° - (70° + 100° + 100°) \quad \text{(∠ sum of quad.)}\\ &= 360° - 270°\\ &= 90°\end{aligned}$$

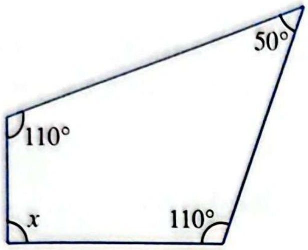
Exercise 6F

1. Find the value of the unknown angle marked in each of the following quadrilaterals. The figures shown are not drawn to scale.

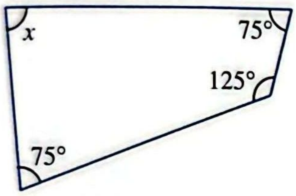
(a)



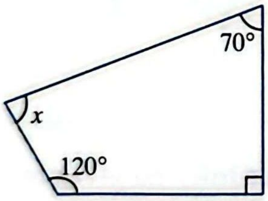
(b)



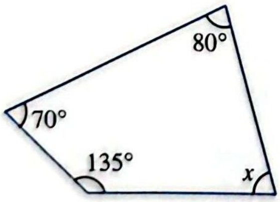
(c)



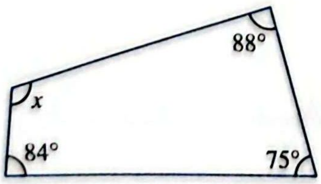
(d)



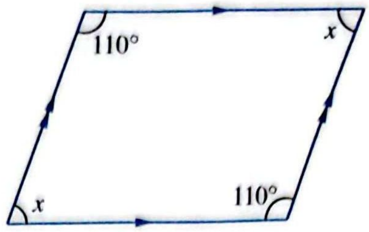
(e)



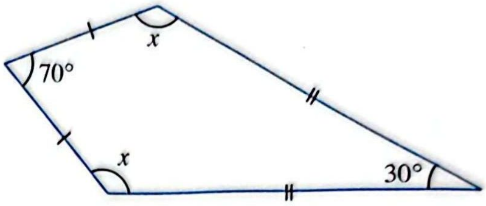
(f)



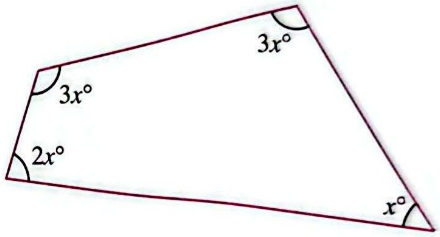
(g)



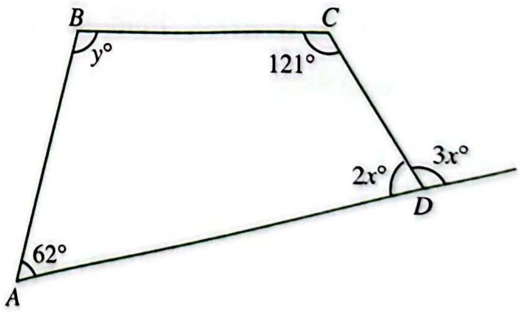
(h)



2. Find the value of x.

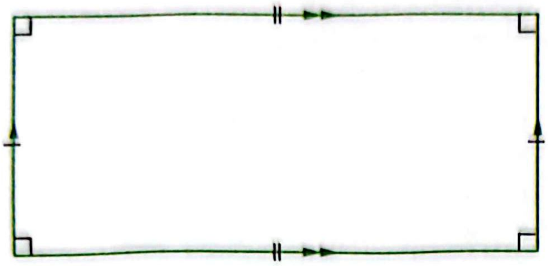


3. Given the angles shown in the diagram, find the values of
   (a) x,
   (b) y.

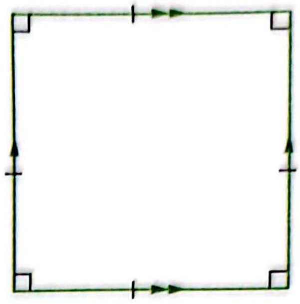


6.4.2 Properties of Special Quadrilaterals

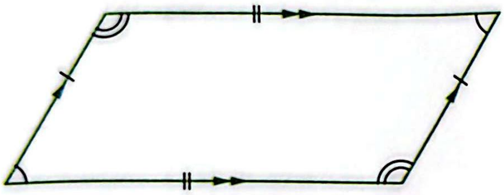
In this section, we will be studying more about the properties of the six special types of quadrilaterals shown below.



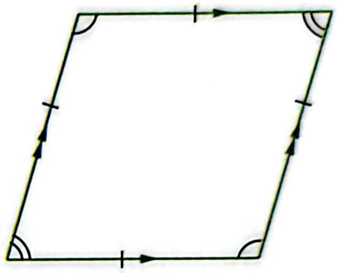
rectangle



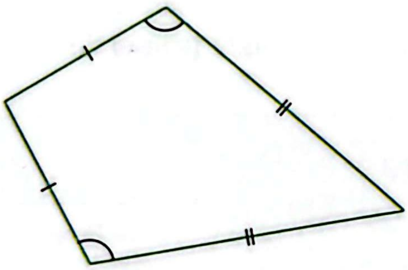
square



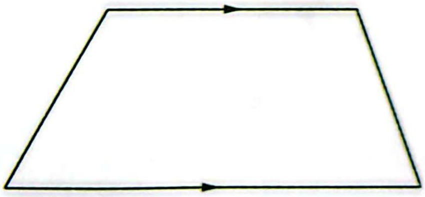
parallelogram



rhombus



kite



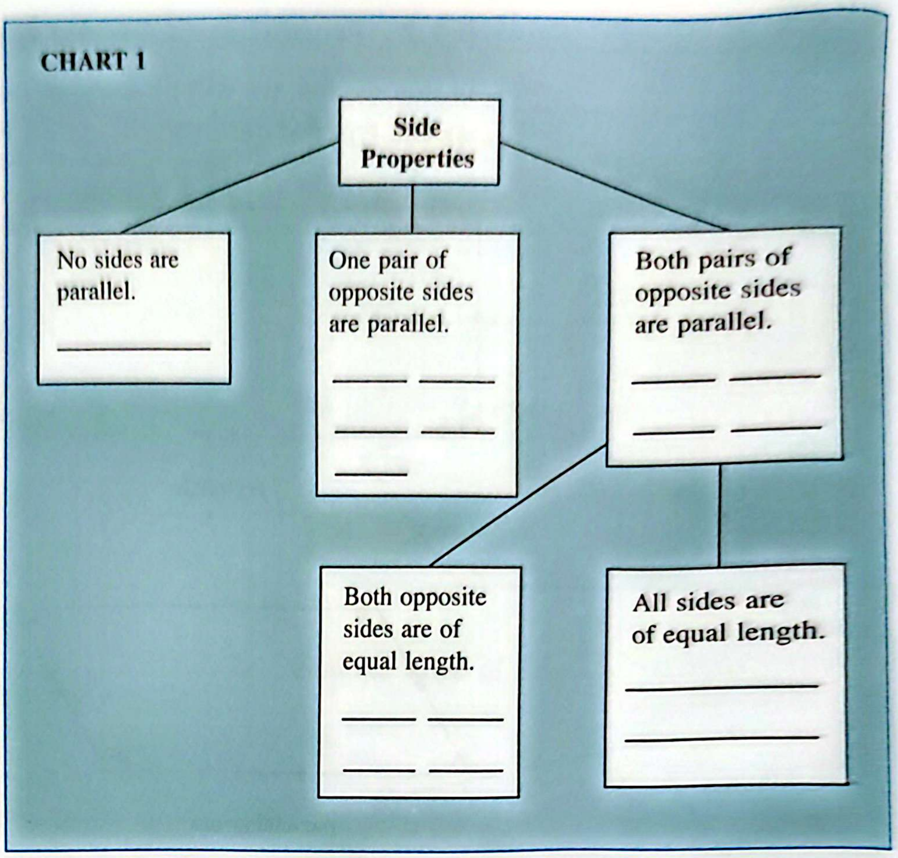
trapezium

Activity

To study the properties of special quadrilaterals.

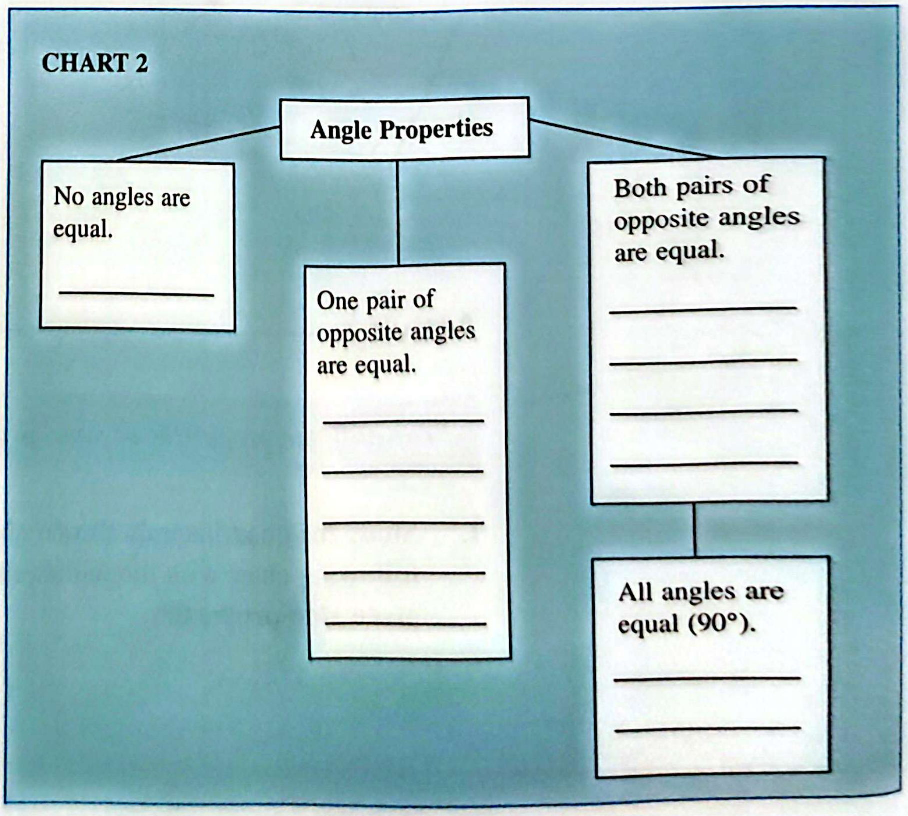
1. Study the quadrilaterals shown above. Then, copy and complete the following chart with the names of the quadrilaterals that satisfy the given **side properties**.

Classifying



Classifying

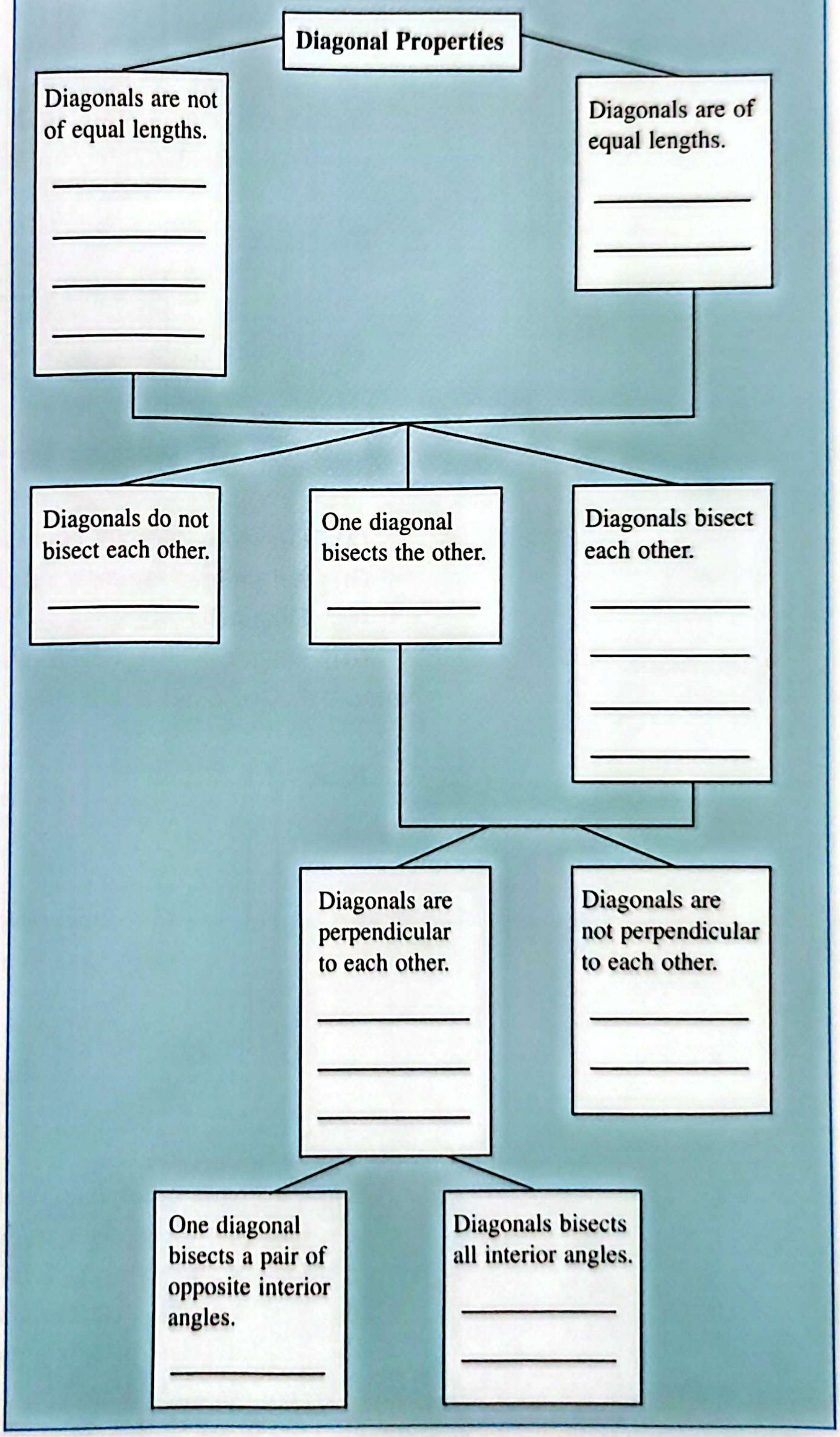
2. Study the quadrilaterals shown on the previous page. Then, copy and complete the following chart with the names of the quadrilaterals that satisfy the given **angle properties**.



Classifying

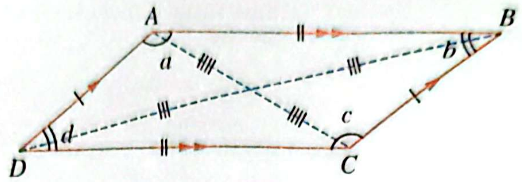
3. (a) Trace all the six quadrilaterals on page 197 onto a piece of paper.
(b) For each quadrilateral, draw its two diagonals.
Study the diagonals formed in each of the quadrilaterals. Then, copy and complete the following chart with the names of the quadrilaterals that satisfy the given **diagonal properties**.

CHART 3



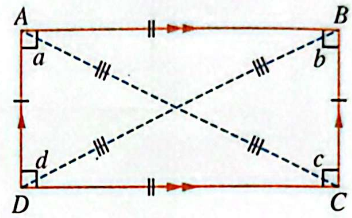
The properties of the six types of special quadrilaterals can be summarised as follows:

1. **Parallelogram:**



(a) Opposite sides are equal and parallel.
(b) Opposite angles are equal.
(c) Diagonals bisect each other.

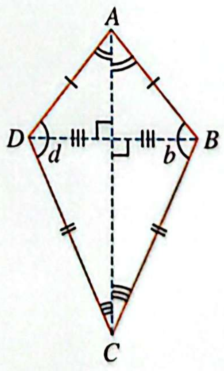
2. **Rectangle:**



(a) Opposite sides are equal and parallel.
(b) All interior angles are right angles.
(c) Diagonals are equal.
(d) Diagonals bisect each other.

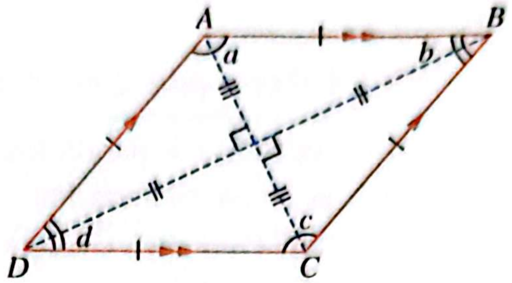
Note: A rectangle has all the properties of a parallelogram.

3. **Kite:**



(a) Two pairs of adjacent sides are equal.
(b) One pair of opposite angles are equal.
(c) Diagonals intersect each other at right angles.
(d) One diagonal bisects the other.
(e) One diagonal bisects a pair of opposite angles.

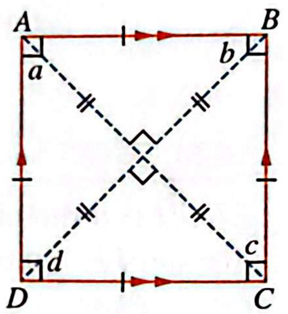
4. **Rhombus:**



(a) All sides are equal.
(b) Opposite sides are parallel.
(c) Opposite angles are equal.
(d) Diagonals bisect each other at right angles.
(e) Diagonals bisect all interior angles.

Note: A rhombus has all the properties of a parallelogram.

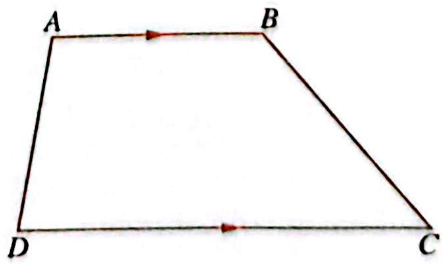
5. **Square:**



(a) All sides are equal.
(b) Opposite sides are parallel.
(c) All interior angles are right angles.
(d) Diagonals are equal.
(e) Diagonals bisect each other at right angles.
(f) Diagonals bisect all interior angles.

Note: A square has all the properties of both a parallelogram and a rhombus.

6. **Trapezium:**



One pair of opposite sides are parallel.

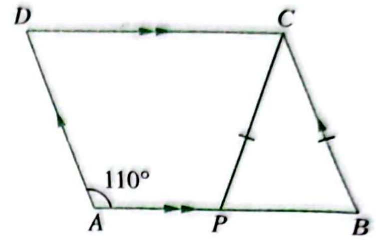
With the properties of special quadrilaterals that we have just learned, we can solve some problems involving angles in such quadrilaterals.

Example 12

ABCD is a parallelogram. *P* is a point on *AB* such that $\angle PAD = 110°$ and $PC = BC$. Calculate

(a) $\angle ABC$,

(b) $\angle DCP$.



Solution

(a) $\angle ABC = 180° - 100°$ (int. ∠s, // lines)

$= 70°$

(b) $\angle BPC = 70°$ (base ∠s of isos. △)

$\angle DCP = \angle BPC$ (alt. ∠s, // lines)

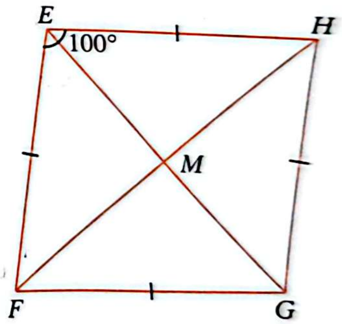
$= 70°$

Example 13

EFGH is a rhombus in which the diagonals *EG* and *FH* intersect at *M*. Find

(a) $\angle FEM$,

(b) $\angle EFM$.



Solution

(a) $\angle FEM = \dfrac{100°}{2}$ (Diagonal *EG* bisects $\angle FEH$)

$= 50°$

Can you find $\angle EFM$ by another method? Try it on your own.

(b) $\angle EMF = 90°$ (diagonals of a rhombus bisect each other at rt. ∠s)

$\therefore \quad \angle EFM = 180° - 90° - 50°$ (∠ sum of △)

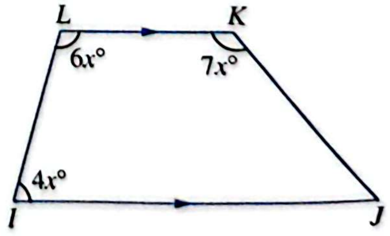
$= 40°$

Example 14

IJKL is a trapezium in which *LK* is parallel to *IJ*. Calculate

(a) the value of *x*,

(b) $\angle IJK$.



Solution

(a) $4x° + 6x° = 180°$ (int. ∠s, // lines)

$10x° = 180°$

$x° = \frac{180°}{10}$

$x° = 18°$

$\therefore \quad x = 18$

(b) $\angle JKL = 7x°$

$= 7 \times 18°$

$= 126°$

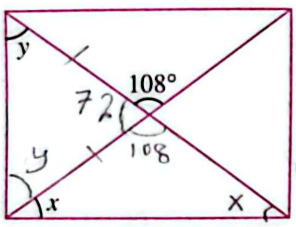
$\therefore \quad \angle IJK = 180° - 126°$ (int. ∠s, // lines)

$= 54°$

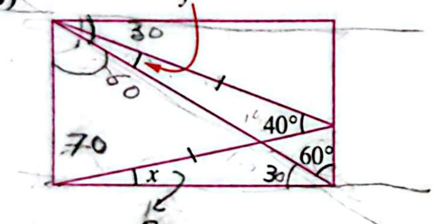
Exercise 6G

1. For each of the rectangles given below, calculate the unknown angles marked x and y.

(a)



(b)

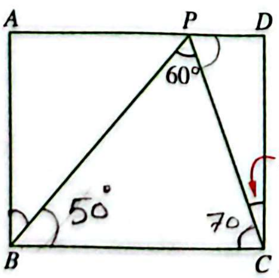


2. $ABCD$ is a rectangle in which $\angle BPC = 60°$ and $\angle PCD = 20°$.

Calculate

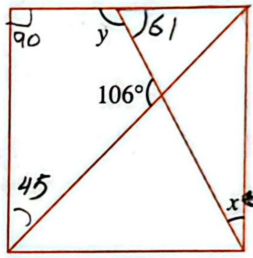
(a) $\angle CPD$,

(b) $\angle ABP$.

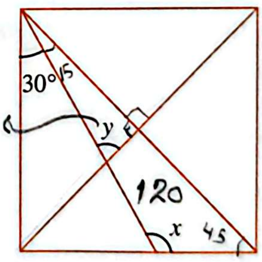


3. For each of the squares given below, calculate the unknown angles marked x and y.

(a)



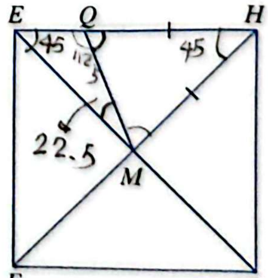
(b)



4. $EFGH$ is a square in which the diagonals intersect at M. Q is a point on EH such that $HQ = HM$. Calculate

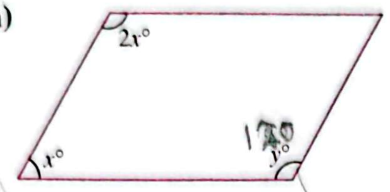
(a) $\angle HQM$,

(b) $\angle EMQ$.

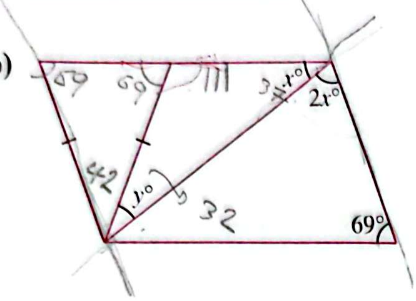


5. For each of the parallelograms given below, calculate the values of x and y.

(a)



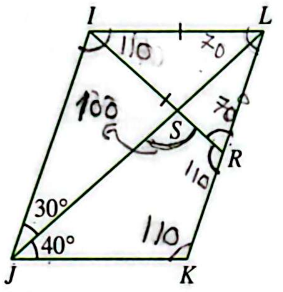
(b)



6. *IJKL* is a parallelogram in which *R* is a point on *LK* such that *IR* = *IL*. *IR* intersects the diagonal *JL* at *S*. Given that $\angle IJL = 30°$ and $\angle KJL = 40°$, calculate

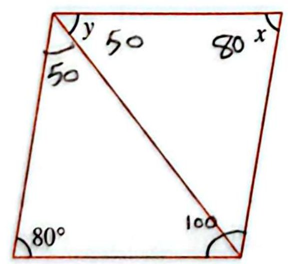
(a) $\angle IRL$,

(b) $\angle JSR$.

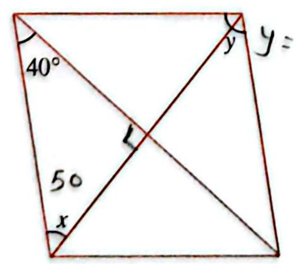


7. For each of the rhombuses given below, calculate the unknown angles marked x and y.

(a)

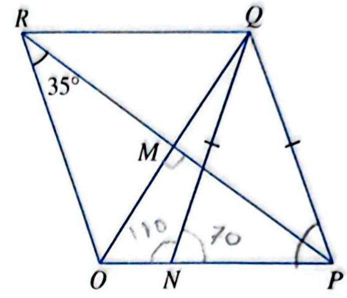


(b)



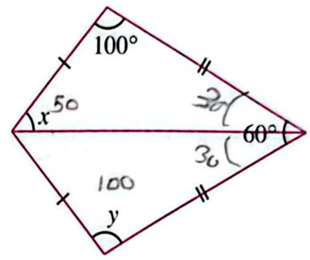
8. *OPQR* is a rhombus in which the diagonals intersect at *M*. *N* is a point on *OP* such that *QN* = *QP*. Given that $\angle ORP = 35°$, calculate

(a) $\angle OPQ$, (b) $\angle ONQ$.

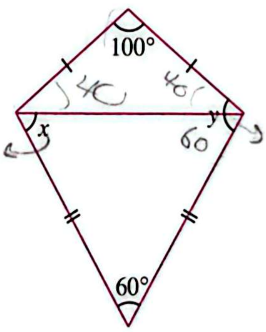


9. For each of the kites given below, calculate the unknown angles marked x and y.

(a)

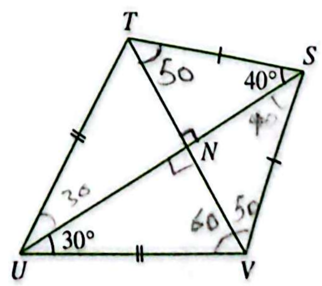


(b)



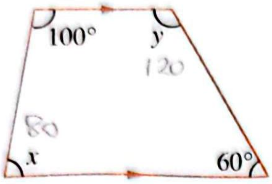
10. *STUV* is a kite in which the diagonals intersect at *N*. Calculate

(a) $\angle STN$, (b) $\angle SVU$.

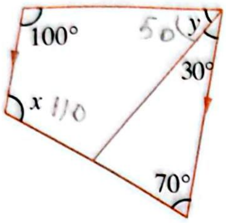


11. For each of the trapeziums given below, calculate the unknown angles marked x and y.

(a)



(b)



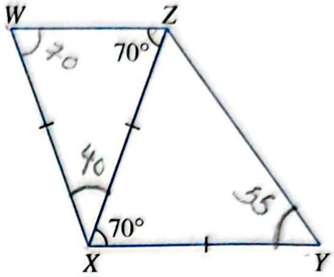
12. WXZ and XYZ are isosceles triangles in which $WX = YX = ZX$ and $\angle WZX = \angle YXZ = 70°$.

(a) Calculate

(i) $\angle WXZ$,

(ii) $\angle XYZ$.

(b) State the name of this special type of quadrilateral.



6.5 Constructing Quadrilaterals

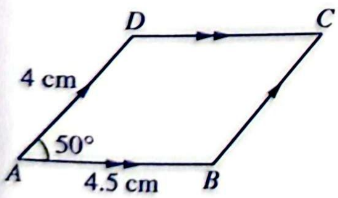
In this section, you will learn how to construct quadrilaterals when given certain measurements.

Let us take a look at some examples of how we can construct quadrilaterals with given measurements using some of these tools: a pair of compasses, a ruler, a set-square and/or a protractor.

Example 15

Construct a parallelogram $ABCD$ in which $\angle DAB = 50°$, $AB = 4.5$ cm and $AD = 4$ cm.

Sketch:



Solution

Step 1 Draw a line 4.5 cm long. Label it AB.

Step 2 At A, use a protractor to draw an angle of 50°.

Step 3 With centre A and radius 4 cm, draw an arc to cut this ray. Label this point of intersection as D.